



# Theory of directional pulse propagation: detailed calculations

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I construct combined electric and magnetic field variables which independently represent energy flows in the forward and backward directions respectively, and use these to re-formulate Maxwell's equations. The emphasis is on detailed calculations, with a more general overview being published in Phys. Rev. A72 and arXiv. These directional variables enable us to not only judge the effect and significance of backward-travelling field components, but also to discard them when appropriate. They thereby have the potential to simplify numerical simulations, leading to potential speed gains of up to 100% over standard FDTD or PSSD simulations. These field variables are also used to derive both envelope equations useful for narrow-band pulse propagation, and a second order wave equation. Alternative definitions are also presented, along with their associated wave equations.

This report should be read along with the paper Phys.Rev.A72, 063807 (2005) “*Theory of directional pulse propagation*”, by P. Kinsler, S.B.P. Radnor, G.H.C. New, for proper context.

This document is primarily intended as a complete (as possible) record of the calculational steps that were necessarily abbreviated (or omitted) from that published work; it also contains a great many other relevant calculations along with some speculation. It is an edited version of a longer document from which on-going work has been excised; and, as a “work in progress”, despite my best efforts, may contain occasional mistakes. Please contact me if you have any comments, corrections or queries.

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## I. INTRODUCTION

We define field variables with directional properties, and use them to show how to optimise both our understanding of pulse propagation and numerical simulations. Simple plane-polarized versions of these were originally introduced by Fleck [1], using  $\epsilon^{1/2}E_x \pm \mu^{1/2}H_y$ . However, these were little used and the bulk of the paper primarily addressed generating ultra-short pulses by Q-switched lasers, discussing numerical techniques and simulation results. Here we derive wave equations and present results using generalised versions of these field variables. In the course of our investigations, we use these variables to illuminate several interesting features of  $z$ -propagated optical pulse simulations which are usually overlooked in a “forward-only” approximation.

The main novelty of these new field variables is that they correspond to forward and backward directed energy fluxes; and are constructed from combinations of the electric and magnetic fields, along with the properties of the propagation medium.

I also derive wave equations describing the propagation of these field components, in fully vectorised and plane-polarized versions. In addition to these exact re-expressions of Maxwell’s equations, I also present second order and envelope-based wave equations for  $G^\pm$ , in both stationary and moving frames. A further advantage of this approach is that it is as easy to include magneto-optic effects as electro-optics ones (i.e. dispersion and nonlinearity). For example, I will later show that the plane polarized first order wave equations describing the  $z$ -propagation of  $G^\pm$  in the spectral ( $\omega$ ) domain are –

$$\partial_z G^\pm = \mp i\omega\alpha_r\beta_r G^\pm \mp \frac{i\omega\beta_r}{2}\alpha_c * [G^+ + G^-] - \frac{i\omega\alpha_r}{2}\beta_c * [G^+ - G^-], \quad (1.0.1)$$

where  $\alpha_r = \sqrt{\epsilon_L}$ ,  $\beta_r = \sqrt{\mu_L}$  contain the linear response of the medium, and  $\alpha_c$  contains nonlinearity that affects the electric field, and  $\beta_c$  is its magnetic counterpart. Using these forward and backward directed  $G^\pm$  variables has three main advantages:

- The forward directed field variable  $G^+$  is the appropriate physical choice to use when studying the forward propagating part of the field – as is the case for most pulse propagation investigations.
- Because we can choose to simulate only the forward directed field variable  $G^+$ , we have half the calculations to do, and hence gain a speed advantage. By using a  $z$ -propagated PSSD algorithm[4], we also get a fast and flexible treatment of dispersive and nonlinear effects; hence our simulations significantly outperform standard FDTD approaches.
- Most other approaches (even the recent [5]) assume the backward propagating parts of the field are negligible. Since our  $G^-$  describes the backward directed part, we have a clear and physically appropriate basis on which to calculate it, and hence can (when appropriate) rigorously justify the common forward-only approximation.

We can do even better by changing to a co-moving frame. This simplifies the  $G^+$  propagation, by removing the frequency-like oscillations, but has the price of making the  $G^-$  oscillations faster. However, in the limit where we decide to approximate by ignoring the backward propagating  $G^-$  part of the field, this price becomes irrelevant.



### A. Why am I doing this? An anecdotal history

Fleck's definitions were brought to my attention by Geoff New shortly after I started working with at Imperial in 2001. Because they gave rise to additional  $d/dt$  terms in the polarization, when compared to standard forms, he wondered whether there was a link with the Brabec and Krausz derivation[8] of similar-looking corrections to an envelope propagation equation. Although it turns out that this was not the case, I continued working on the idea. At first I focussed on generating a second order wave equation of  $G^\pm$ , in analogy to the one usually presented for the  $E$  field. Although I generalized my GFEA derivation [7, 9] to include Fleck-like terms, there seemed no practical use for the technique.

In 2002 a PhD student in our group (JCA Tyrrell) started working on Maxwell's equations solvers, and applying them to pulse propagation in nonlinear media. He developed the PSSD method [4] for solving Maxwell's equations, where the field is held as a function of time and propagated forward in space. This is common in envelope-based nonlinear optics simulations, but had not been applied to full Maxwell's solvers before. This got me thinking about representing Maxwell's equations using  $G^\pm$ , and put the project on track to being of practical use, although there were many outstanding questions on interpretation of the fields and the wave equations. While trying to clarify these issues, I became aware of work by Kolesik and co-workers [5, 6]), who projected out Maxwell's equations into a directional form. Their treatment motivated me on to vectorize the definitions, and get a PhD student (SBP Radnor) to start on simulations. As we worked on the paper eventually published as [2], I further generalised the definitions to include the longitudinal parts of the fields, fully developed the variant forms of the variables, and derived envelope versions of the wave equations.

As it stands, these kinds of directional wave equations, either using my  $G^\pm$  variables, or the Kolesik et.al. approach make traditional envelope theories based on the second order wave equation for  $E$  utterly redundant. This is true even for the generalised ones such as [7, 8] which incorporate wideband corrections. Directional wave equations simple provide propagation equations without introducing the many approximations inherent in theories based on second order forms, as are detailed exaustively in [7, 9]. This is treated in more detail in my report "*Field and envelope methods in nonlinear optics*".

I am occasionally asked why I picked  $G^\pm$  instead of (e.g.)  $F^\pm$  to name these directional fields – after all,  $F$  follows alphabetically from the electric field (letter)  $E$ , and it is the first letter of Fleck's surname, so it might seem the more obvious choice. Unfortunately I can't remember why; but there was a reason why I passed over  $F$ , I think something to do with some other  $F$  I was using at the time. I should have picked  $K^\pm$ , anyway.

### B. An alternative approach

#### 1. Kolesik, Moloney, and Mlejnek; PRL 89, 283902 (2002); and projection operators

Kolesik et.al. [6] also use not-dissimilar combinations of field variables. They introduce projection operators for the forward and backward parts of the field in their eqn (KMM 1), which I rewrite slightly using  $\vec{u} = \vec{k}/|\vec{k}|$ :

$$\mathcal{P}^\pm \vec{D}(\vec{k}) = \frac{1}{2} \left[ \vec{D}(\vec{k}) \mp \text{sgn}(kz) \frac{k}{\omega(k)} \vec{u} \times \vec{H}(\vec{k}) \right] \quad (1.2.1)$$

$$\mathcal{P}^\pm \vec{H}(\vec{k}) = \frac{1}{2} \left[ \vec{H}(\vec{k}) \pm \text{sgn}(kz) \frac{\omega(k)}{k} \vec{u} \times \vec{D}(\vec{k}) \right]. \quad (1.2.2)$$

They then use these, with Maxwell's equations to generate their UPPE in eqn (KMM 7):

$$\partial_t \vec{D}_f(\vec{k}) = -i\omega(k) \vec{D}_f(\vec{k}) + \frac{i}{2} \omega(k) \left[ \vec{P}_{NL}(\vec{D}, \vec{k}) - \vec{u} \left( \vec{u} \cdot \vec{P}_{NL}(\vec{D}, \vec{k}) \right) \right]. \quad (1.2.3)$$

There are three main difference compared with my generalised Fleck-style  $G^\pm$  approach. They:

1. describe things in terms of projection operators,
2. project out *both* a directional  $D$  and a directional  $H$ ,
3. construct their (effective) directional variables in a different (but related) way.



Another difference is that, when implementing a solution, they keep the *time derivatives* on the LHS, and manipulate the  $\nabla \times$  terms on the RHS; so their UPPE gives a time derivative of the field, and so their fields are propagated through time. This contrasts with my approach which keeps the  $\nabla \times$  on the LHS, and manipulates the time derivatives – hence my fields are propagated through space. Their choice is (most probably) why they choose to work with the displacement field  $D$  rather than  $E$ . In the time domain, the use of  $D$  avoids the need for the  $D = \epsilon * E$  convolution of the field with the permittivity, at least on the time-derivative side of the Maxwell's equations. They (presumably) remove the space derivatives resulting from the  $\nabla \times$  terms when going into the spatial frequency ( $k$ ) domain, leading to the appearance of  $K$  terms on the RHS.

However it is not so easy to avoid complication when solving Maxwell's equations. Their use of  $D$  simplifies the time derivative side, but there is a cost to pay on the space derivative ( $\nabla \times$ ) side, where they need to reconstruct  $E$  from  $D$  (see eqn (DMN 2)), and similarly evaluate the nonlinearity (see eqn (DMN 3)). Even then, just prior to eqn (DMN 7), Kolesik et.al. explain how they go into the frequency domain anyway! As an aside, they might easily have used  $B$  and not  $H$  in their equations, and so included the possibility of magnetic media in their model.

In contrast, I leave the space derivative ( $\nabla \times$ ) sides alone, and work with the time derivative side. This has the advantage that when I go into the frequency domain, the convolution becomes a simple product, and I do not have to resort to reconstructions of  $E$  and  $P_{NL}$ . It is interesting to compare the “ $\partial_t$ ” UPPE with my similar-looking “ $\partial_z$ ” eqn (4.1.8) that follows later, where they neglect to introduce the advantages of a co-moving frame (cf my eqn(4.3.14)). See also III G, where I attempt a conversion between my formulation and theirs.

**SUMMARY:** Maxwell-like theory with combined  $G$ -like variables, but no co-moving frame. Temporal propagation only.

## 2. Kolesik and Moloney; PRE **70**, 036604 (2004).

This work [5] is an extension of the method proposed in [6], and contains a variety of applications. It discusses both  $t$  and  $z$  propagated approaches, and includes decomposition into transverse modes. They develop an envelope version of their wave equation, but do not discuss using multiple envelopes. In general they only solves for  $E$ , not both  $E$  and  $H$  separately, there is also no examination (or calculation) of the significance of backward propagating fields. They separate the polarization  $P$  from displacement  $D$ ; and includes current terms. See also III G, where I attempt a conversion between my formulation and theirs.

### C. Moving frames

In Fidel et.al. 1997 [10] "Hybrid Ray FDTD moving window approach to pulse propagation", they look at a variety of moving window and moving frame approaches to solving for pulse propagation with Maxwell's equations using FDTD. They do introduce a co-moving frame, and derive the appropriate discretized Maxwell's equations; but they do not use combined  $E$  and  $B$  variables as done by Fleck, and as done in the following sections here.

**SUMMARY:** Maxwell-like theory with co-moving frame, but no combined  $G$ -like variables.

### D. Beltrami Fields, Photon Wave Functions: $\vec{E} + i\vec{B}$

The  $\vec{G}^\pm$  fields I construct here bear a superficial resemblance to Beltrami variables [11, 12, 13, 14], which are defined along the lines of  $\vec{Q} = \sqrt{\epsilon}\vec{E} + i\sqrt{\mu}\vec{H}$ . However, they differ in two important respects. First, for a given Beltrami  $\vec{Q}$  you immediately know what the component  $\vec{E}$  and  $\vec{H}$  fields are, whereas you require both  $\vec{G}^\pm$  to do the same. Secondly,  $\vec{Q}$  does not assume any preferred direction, whereas the construction of  $\vec{G}^\pm$  requires a direction  $\vec{u}$  to be chosen. It's interesting to wonder whether the  $E \pm iH$  approaches might generalise to something like  $E \pm e^{i\phi}H$ . However, since  $E \pm iH$  is unique for a given  $E$  and  $H$ ; and  $E \pm H$  is not unique without both components, so it's not obvious how it might work.

Note that some discussions of Beltrami fields refer to "force free" systems, where  $\nabla \times B = \alpha B$ , which are not (as



far as I can see) applicable to an optics regime.

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## II. THE MATERIAL PROPERTIES: REFERENCE AND CORRECTION PARAMETERS

Maxwell's equations describe propagation of the EM field through a medium described by permittivity  $\epsilon$  and permeability  $\mu$ . In addition, it is common to add in a “polarization” term  $\vec{P}$ , which is used to relate the displacement field  $\vec{D}$  to  $\vec{E}$  in the usual way:  $\vec{D} = \epsilon * \vec{E} + \vec{P}$ . In my treatment, I hide this polarization inside the definition of  $\epsilon$  (writing instead just  $\vec{D} = \epsilon * \vec{E}$  for the same system); similarly any magnetic polarization would be hidden inside  $\mu$  and not written out separately. Note that I do make a split of the total permittivity (and also the permeability) into two pieces, a “reference” part and a “correction” part, and these will not necessarily be the same as the traditional division.

Note 1: I use the “ $*$ ” notation for convolutions  $a(t) * b(t) = \int a(t - t')b(t')dt'$ .

Note 2: Quantities in the frequency domain are indicated by tildes.

### A. Reference and correction contributions

We first denote the linear response of the medium to be  $\epsilon_L$ , and its nonlinear response to be  $\epsilon_{NL}$ . The nonlinear term may include some time response, such as e.g.  $\epsilon_{NL}(t) = \int \epsilon_3(\tau)E(t - \tau)^2d\tau$ . The effect of the total permittivity can be expressed in either the time domain or the frequency domain (indicated by tildes). It is

$$\epsilon(t) * E(t) = \epsilon_L(t) * E(t) + \epsilon_{NL}(t)E(t) \quad (2.1.1)$$

$$\text{or} \quad \tilde{\epsilon}(\omega) * \tilde{E}(\omega) = \tilde{\epsilon}_L(\omega)\tilde{E}(\omega) + \tilde{\epsilon}_{NL}(\omega) * \tilde{E}(\omega). \quad (2.1.2)$$

Similar expressions can be written down for the case of magnetic dispersion or magnetic nonlinearities,

$$\mu(t) * E(t) = \mu_L(t) * E(t) + \mu_{NL}(t)E(t) \quad (2.1.3)$$

$$\text{or} \quad \tilde{\mu}(\omega) * \tilde{E}(\omega) = \tilde{\mu}_L(\omega)\tilde{E}(\omega) + \tilde{\mu}_{NL}(\omega) * \tilde{E}(\omega). \quad (2.1.4)$$

The definitions of  $G^\pm$  (and their generalized vector counterparts  $\vec{G}^\pm$ , introduced below) depend on the properties of the propagation medium through the permittivity  $\epsilon$  and permeability  $\mu$ . In principle it might seem attractive to define  $\vec{G}^\pm$  using the exact values of  $\epsilon, \mu$  (including the nonlinearity), but we will usually want to be able to uniquely reconstruct the fields  $E$  and  $H$  from our new fields. We do this by including as much as possible of the dispersive properties in the reference parameters (and hence the definitions of  $\vec{G}^\pm$ ), leaving nonlinear properties (and potentially some residual dispersion) for correction terms. For example, if the true dispersion was multi-valued or zero in places,



it is possible to define the reference to include an approximation to the true dispersion without those inconvenient properties, and use the correction term to compensate.

The frequency domain is the starting point for calculating the necessary parameters. This is because the formulae are simplest in the  $\omega$  domain, where any linear response is described using products rather than convolutions. This will mean that we calculate the time-domain parameters by (back) Fourier transforming the frequency domain ones. The frequency domain definitions are

$$\tilde{\epsilon} * \tilde{E} = \tilde{\epsilon}_r(\omega) \tilde{E}(\omega) + \tilde{\epsilon}_c(\omega) * \tilde{E} \quad (2.1.5)$$

$$= \tilde{\alpha}_r^2(\omega) \tilde{E}(\omega) + \tilde{\alpha}_r(\omega) \tilde{\alpha}_c(\omega) * \tilde{E}(\omega), \quad (2.1.6)$$

$$\tilde{\mu} * \tilde{H} = \tilde{\mu}_r(\omega) \tilde{H}(\omega) + \tilde{\mu}_c(\omega) * \tilde{H}(\omega) \quad (2.1.7)$$

$$= \tilde{\beta}_r^2(\omega) \tilde{H}(\omega) + \tilde{\beta}_r(\omega) \tilde{\beta}_c(\omega) * \tilde{H}(\omega), \quad (2.1.8)$$

where the correction parameters  $\tilde{\epsilon}_c$  and  $\tilde{\mu}_c$  represent the discrepancy between the true values and the reference. The smaller these correction terms are, the better the match. By using these frequency dependent parameters in the generalized definitions of  $G^\pm$ , we will be able to propagate pulses using only the  $G^+$  variable, a gain in both mathematical simplicity and computational speed. Note that since the definitions of  $G^\pm$  depend (roughly speaking) on the square roots of  $\tilde{\epsilon}$  and  $\tilde{\mu}$ , we have introduced the  $\tilde{\alpha}$  and  $\tilde{\beta}$  parameters, which will feature prominently (along with their time domain counterparts  $\alpha, \beta$ ), in the generalized definitions of  $\vec{G}^\pm$  that follow.

To recap, I split the medium properties into two parts:

**First**, there are the “reference” parts  $\epsilon_r$  and  $\mu_r$ . These are the contributions from the material properties which will be incorporated into the definition(s) of  $G^\pm$ . These will only contain *linear* response terms, to guarantee we can reconstruct  $E, H$  from  $G^\pm$

**Second**, there are the “correction” parts  $\epsilon_c(t)$  and  $\mu_c(t)$ , which will contain the difference between the chosen reference and the true behaviour of the material. These will only contain any *nonlinear* response terms, but may also contain an residual linear response also.

We might also choose to split the correction terms up into more pieces – e.g. a dispersive part, and a nonlinear part: e.g.,

$$\tilde{\alpha}_c * \tilde{E} = \tilde{\alpha}_c^D \tilde{E} + \tilde{\alpha}_c^{NL} * \tilde{E}. \quad (2.1.9)$$

## B. Nonlinearity

Since it is usually impractical to include nonlinearities in the reference parameters, these will normally appear in the correction terms  $\epsilon_c, \mu_c$ . As an example, consider a  $n$ -th order (electric) nonlinearity, in which case  $\epsilon_c(t) = \chi^{(n)}(t) * E(t)^{n-1}$ , and

$$\tilde{\alpha}_r(\omega) \tilde{\alpha}_c(\omega) = \mathcal{F}[\chi^{(n)}(t) * E(t)^{n-1}] \quad (2.2.1)$$

$$\tilde{\alpha}_c(\omega) = [\tilde{\alpha}_r(\omega)]^{-1} \cdot \mathcal{F}[\chi^{(n)}(t) * E(t)^{n-1}], \quad (2.2.2)$$

$$\alpha_c(t) = \mathcal{F}^{-1}\left\{[\tilde{\alpha}_r(\omega)]^{-1} \cdot \tilde{\chi}^{(n)}(\omega) \cdot \mathcal{F}[E(t)^{n-1}]\right\}, \quad (2.2.3)$$

$$\text{where (see later)} \quad E(t) = \frac{1}{2} \left\{ \mathcal{F}^{-1}[\tilde{\alpha}_r^{-1}] \right\} * [G^+ + G^-] \quad (2.2.4)$$

where  $\mathcal{F}[\dots]$  is the Fourier transform (FT) from time to frequency, and  $E(t)$  can be found from eqn. (3.3.3). If the reference parameters  $\tilde{\alpha}_r$  contain dispersion (which will be the typical case), we can see from eqn. (2.2.2) that this will make  $\tilde{\alpha}_c(\omega)$  dispersive even if  $\chi^{(n)}$  is instantaneous. In the case of an instantaneous nonlinearity, this adds more computational work (an extra two FTs), although for non-instantaneous ones we needed the FTs anyway. If the nonlinearity is instantaneous *and* the reference parameters are non-dispersive, we have simply  $\alpha_c^{NL}(t) = \alpha_r^{-n} \cdot \chi^{(n)} \cdot 2^{-n+1} [G^+ + G^-]^{n-1}$ .

In practise, therefore, calculation of an instantaneous nonlinear term will involve calculating  $\chi^{(n)} E(t)^n$ , fourier transforming (denoted by  $\mathcal{F}$ ) the result, then dividing it by the (possibly frequency dependent) reference parameter  $\alpha_r$ . If the reference parameter  $\alpha_r$  is a constant, then the fourier transform is redundant and the nonlinear step can be handled while remaining entirely in the time domain. Note that there is no way we can avoid convolutions if the reference  $\alpha_r$  contains any frequency dependence – even if our nonlinearity is instantaneous. This is because to calculate  $\alpha_c$  I have to remove the reference  $\alpha_r$  from  $\tilde{\epsilon}_c$ ; since  $\alpha_c(\omega) = \tilde{\epsilon}_c(\omega)/\alpha_r(\omega)$ .



### III. DEFINITIONS OF $\vec{G}^\pm$

Here I go significantly further than Fleck[1] in two respects: I *vectorise the definitions*, and *allow for dispersive effects*. An important addition is the inclusion of a longitudinal field component, which is required in order to retain a full vector description of the EM field. Originally I tried to generate a motivation for the directional  $G^\pm$  fields described above by taking Maxwell's equations and moving to a co-moving frame. However, it seems now that while they certainly seem to describe parts of the field propagating in opposite directions, they do not seem tied to any particular choice of frame.

After a brief discussion of what Fleck did, I define a vector form  $\vec{G}^\pm$  of the Fleck-style fields  $G^\pm$ . This requires the definition of a “direction of propagation” denoted by the constant unit vector  $\vec{u}$ . I can then achieve the swapping of the transverse components of  $\vec{H}$  (since  $G_x \sim E_x \pm H_y$ ) by using the cross product of  $\vec{u}$  with the magnetic field variable  $\vec{H}$ .

Secondly I use “reference” material parameters which I use to construct my  $G^\pm$  fields. This allows me to distinguish between the parameters used in my construction of  $G^\pm$  from the actual material properties, which may be too complicated to use in that context. Generally, I include all of the linear response of the medium in these reference parameters, which makes it easiest to define  $\vec{G}^\pm$  in the frequency domain. This is because the linear time-response convolution in the time-domain becomes a simple product in the frequency domain. A good (accurate) choice of reference medium for a particular problem will ensure that the contribution due to the backward directed part of the field ( $G^-$ ) is small. In general we would want the reference part to include as much about the material as is practicable, to ensure the smallest possible correction term(s) for propagation.

Note that whether or not the  $\epsilon, \mu$  contain dispersion, we here restrict ourselves to the case where they are *scalar*. Also, I let the *argument* of the vector field quantities tell us whether we are in the time or frequency domain, to avoid too much notational clutter. As a final note, in the following calculations the assumption will be made that the media are bulk, without (implicit) interfaces hidden in  $\epsilon$  or  $\mu$ . This means that  $\partial_z \epsilon = \partial_z \mu = 0$  is assumed.

#### A. Fleck's approach

Fleck defines a relative permittivity  $\epsilon$ , and magnetic permeability  $\mu$  in a dispersionless host medium – it is clear that these are relative parameters because he later defines  $\eta = (\epsilon\mu)^{1/2}$  as the refractive index of the host medium (just after eqn (F1.4b)). He allows for a Conductivity  $\sigma$ ; and a Polarization  $P$  which characterises the active atoms and which may be amplifying or absorbing. The electric field is  $E_x$  and the magnetic field  $B_y = \mu H_y$ . As usual, the speed of light  $c$  is just related to the permittivity and permeability of the vacuum, i.e.  $(\epsilon_0 \mu_0)^{-1/2}$ . Fleck simplifies plane polarised Maxwell field equations, treating them only along the  $z$  direction, resulting in his eqn.(F1.1a,b)

$$\partial_t \frac{\epsilon}{c} E_x = -\partial_z H_y - \frac{4\pi}{c} \partial_t P - \frac{4\pi}{c} \sigma E_x, \quad (3.1.1)$$

$$\partial_t \frac{\mu}{c} H_y = -\partial_z E_x. \quad (3.1.2)$$

Then Fleck defines new combined EM field variables  $E^\pm$  in eqn.(F1.3), but I write them  $G^\pm$  to avoid confusion with the  $E$  field variables.

$$E^\pm \equiv G^\pm = \epsilon^{1/2} E_x \pm \mu^{1/2} H_y. \quad (3.1.3)$$

Because of the relative phases of  $E$  and  $H$  in these definitions, if the two fields  $G^+$  and  $G^-$  are constants multiplied by a standard exponentially oscillating carrier wave, they are associated with an energy flux (i.e. Poynting vector) in opposite directions along the  $z$  axis; this will presumably also hold nearly true when replacing the constant amplitude with “slowly varying” pulse envelopes. See also section VIII.

Fleck uses his definitions to obtain equations of motion for his field variables  $G^+$  and  $G^-$  ( $E^\pm$ ), recorded in his eqn.(F1.4). They are

$$\frac{\eta}{c} \partial_t G^+ + \partial_z G^+ = -\frac{4\pi \mu^{1/2}}{c} \partial_t P - \frac{2\pi \sigma}{c} \left(\frac{\mu}{\epsilon}\right)^{1/2} (G^+ + G^-), \quad (3.1.4)$$

$$\frac{\eta}{c} \partial_t G^- - \partial_z G^- = -\frac{4\pi \mu^{1/2}}{c} \partial_t P - \frac{2\pi \sigma}{c} \left(\frac{\mu}{\epsilon}\right)^{1/2} (G^+ + G^-). \quad (3.1.5)$$



I now scale eqns.(3.1.4,3.1.5) (Fleck (F1.4a, F1.4b)), setting  $G'^{\pm} = (\eta/c) G^{\pm}$ ,  $P' = (4\pi\mu^{1/2}/c) P$ , and  $\sigma' = (2\pi\sigma/\epsilon)^{1/2}$ . In the following, I will drop the prime ('') marks and use  $G$  for  $G'$  (etc) in order to reduce the visual clutter and make equations easier to read. *For this subsection only*, from this point onwards each occurrence of  $G^{\pm}$ ,  $P$ , or  $\sigma$  should be understood to refer to the prime variables  $G'^{\pm}$ ,  $P'$  &  $\sigma'$  – unless explicitly stated otherwise. The scaled equations are

$$\partial_t G^+ + \partial_z G^+ = -\partial_t P - \sigma (G^+ + G^-), \quad (3.1.6)$$

$$\partial_t G^- - \partial_z G^- = -\partial_t P - \sigma (G^+ + G^-). \quad (3.1.7)$$

## B. Alternative forms

The “standard” definition for directional fields handles the electric field  $E$  most easily; although rather non-intuitively we will find that it makes it hard to include longitudinal  $E$  components. Fortunately, other definitions are possible, and these follow the standard “Primary  $\vec{E}$ ” definition presented next in subsection III C, in subsections III D, III E, and III F.

Use of an alternative definition can potentially shift the non-transverse effects to a field variable where it causes less trouble. Note that circularly polarized forms can be generated for all types the same way as is described for primary- $\vec{E}$  type  $\vec{G}^{\pm}$  variables in subsubsection III C 2.

These alternative expressions are represented only in the frequency domain; time domain forms are presented for primary- $\vec{E}$  only; for the other forms it is a simple matter to construct their time domain representation. In the case of dispersionless reference media (i.e. constant  $\epsilon_r(\omega), \mu(\omega)$ ), the time and frequency forms look very similar. The set of alternative definitions was (partly) inspired by PDD suggesting (private communication, 24 February 2003) trying alternative combinations – such as swapping  $D$  for  $E$  and/or  $B$  for  $H$ .

## C. Primary $\vec{E}$ : the standard vector form

This vector definition of directional fields most closely mirrors the simple form suggested by Fleck. For completeness, I will write them down in both time and frequency domains. Note that the use of  $\vec{u} \times \vec{H}$  means that the  $\vec{G}^{\pm}$  will not contain any information about  $\vec{u} \cdot \vec{H}$ , the longitudinal part of  $\vec{H}$  – thus we insist  $\vec{G}^{\pm}$  be magnetically transverse and define a  $\vec{G}^{\circ}$ . The vector fields  $\vec{G}^{\pm}$  are (time domain)

$$\vec{G}^{\pm}(t) = \alpha_r(t) * \vec{E}(t) + \vec{u} \times \beta_r(t) * \vec{H}(t), \quad (3.3.1)$$

$$\vec{G}^{\circ}(t) = [\vec{u} \cdot \beta_r(t) * \vec{H}(t)] \quad (3.3.2)$$

$$\text{so } \alpha_r(t) * \vec{E}(t) = \frac{1}{2} [\vec{G}^+(t) + \vec{G}^-(t)] \quad (3.3.3)$$

$$\text{and } \vec{u} \times \beta_r(t) * \vec{H}(t) = \frac{1}{2} [\vec{G}^+(t) - \vec{G}^-(t)], \quad (3.3.4)$$

or, in the frequency domain, where we can avoid the convolutions

$$\vec{G}^{\pm}(\omega) = \tilde{\alpha}_r(\omega) \vec{E}(\omega) \pm \vec{u} \times \sqrt{\mu_r(\omega)} \vec{H}(\omega), \quad (3.3.5)$$

$$\vec{G}^{\circ}(\omega) = [\vec{u} \cdot \tilde{\beta}_r(\omega) \vec{H}(\omega)], \quad (3.3.6)$$

$$\text{so } \tilde{\alpha}_r(\omega) \vec{E}(\omega) = \frac{1}{2} [\vec{G}^+(\omega) + \vec{G}^-(\omega)] \quad (3.3.7)$$

$$\text{and } \vec{u} \times \tilde{\beta}_r(\omega) \vec{H}(\omega) = \frac{1}{2} [\vec{G}^+(\omega) - \vec{G}^-(\omega)] \quad (3.3.8)$$

$$\text{or } \tilde{\beta}_r(\omega) \vec{H}(\omega) = \frac{1}{2} \vec{u} \times [\vec{G}^+(\omega) - \vec{G}^-(\omega)] + \vec{u} \vec{G}^{\circ}, \quad (3.3.9)$$

since  $\vec{u} \times \vec{u} \times \vec{A} = [\vec{u} \cdot \vec{A}] - \vec{A}$ . In general usage I will remain in the frequency domain, because there I avoid complications due to convolutions (or worse, deconvolutions); in addition the notation is simpler.

Definition: THF – transverse  $H$  (magnetic intensity) field (i.e.  $\vec{u} \cdot \vec{H} = 0$ ).

A derivation of the wave equations using this form is in section IV.



## 1. Divergence

In the normal description of EM, the divergence  $\nabla \cdot \vec{D} = \rho$ . However the picture here is based on  $\vec{E}$ , so we need a way to construct  $\vec{D}$  from  $\vec{E}$ . We follow the following iterative procedure, which should work for the usual case of weak nonlinearities

$$\vec{D}(\omega) = \tilde{\alpha}_r^2 \vec{E}(\omega) + \tilde{\alpha}_r \tilde{\alpha}_c * \vec{E}(\omega) \quad (3.3.10)$$

$$\implies \tilde{\alpha}_r^2 \vec{E}(\omega) = \vec{D}(\omega) - \tilde{\alpha}_r \tilde{\alpha}_c * \vec{E}(\omega) \quad (3.3.11)$$

$$\vec{E}(\omega) = \tilde{\alpha}_r^{-2} \vec{D}(\omega) - \tilde{\alpha}_r^{-1} \tilde{\alpha}_c * \vec{E}(\omega) \quad (3.3.12)$$

$$= \tilde{\alpha}_r^{-2} \vec{D}(\omega) - \tilde{\alpha}_r^{-1} \tilde{\alpha}_c * \left[ \tilde{\alpha}_r^{-2} \vec{D}(\omega) - \tilde{\alpha}_r^{-1} \tilde{\alpha}_c * \vec{E}(\omega) \right] \quad (3.3.13)$$

$$= \tilde{\alpha}_r^{-2} \vec{D}(\omega) - \tilde{\alpha}_r^{-3} \tilde{\alpha}_c * \vec{D}(\omega) + \tilde{\alpha}_r^{-2} \tilde{\alpha}_c * \tilde{\alpha}_c * \vec{E}(\omega) \quad (3.3.14)$$

$$= \tilde{\alpha}_r^{-2} \vec{D}(\omega) - \tilde{\alpha}_r^{-3} \tilde{\alpha}_c * \vec{D}(\omega) + \tilde{\alpha}_r^{-2} \tilde{\alpha}_c * \tilde{\alpha}_c * \left[ \tilde{\alpha}_r^{-2} \vec{D}(\omega) - \tilde{\alpha}_r^{-1} \tilde{\alpha}_c * \vec{E}(\omega) \right] \quad (3.3.15)$$

$$= \tilde{\alpha}_r^{-2} \vec{D}(\omega) - \tilde{\alpha}_r^{-3} \tilde{\alpha}_c * \vec{D}(\omega) + \tilde{\alpha}_r^{-4} \tilde{\alpha}_c * \tilde{\alpha}_c * \vec{D}(\omega) - \tilde{\alpha}_r^{-3} \tilde{\alpha}_c * \tilde{\alpha}_c * \tilde{\alpha}_c * \vec{E}(\omega) \quad (3.3.16)$$

$$= \sum_{i=0}^{\infty} \tilde{\alpha}_r^{-2-i} [-\tilde{\alpha}_c*]^i \vec{D}(\omega) \quad (3.3.17)$$

Thus,

$$\nabla \cdot \vec{E}(\omega) = \nabla \cdot \sum_{i=0}^{\infty} \tilde{\alpha}_r^{-2-i} [-\tilde{\alpha}_c*]^i \vec{D}(\omega) \quad (3.3.18)$$

$$= \sum_{i=0}^{\infty} \tilde{\alpha}_r^{-2-i} [-\tilde{\alpha}_c*]^i \nabla \cdot \vec{D}(\omega) \quad (3.3.19)$$

$$= \tilde{\alpha}_r^{-2} \sum_{i=0}^{\infty} \tilde{\alpha}_r^{-i} [-\tilde{\alpha}_c*]^i \rho(\omega) \quad (3.3.20)$$

$$\text{with the shorthand notation} \quad = \tilde{\alpha}_{\Sigma}^{-2} \rho(\omega) \quad (3.3.21)$$

It is worth noting that in the alternative constructions following this ‘‘Primary  $\vec{E}$ ’’ one, it is far simpler to calculate the divergence. In any case, we can now write the divergence of  $\vec{G}^{\pm}$  as follows

$$\nabla \cdot \vec{G}^{\pm}(\omega) = \nabla \cdot \tilde{\alpha}_r \vec{E}(\omega) \pm \nabla \cdot \vec{u} \times \tilde{\beta}_r \vec{H}(\omega) \quad (3.3.22)$$

$$= \tilde{\alpha}_{\Sigma}^{-2} \tilde{\alpha}_r \nabla \cdot \vec{D} \pm \tilde{\beta}_r \vec{H} \cdot (\nabla \times \vec{u}) \mp \tilde{\beta}_r \vec{u} \cdot (\nabla \times \vec{H}) \quad (3.3.23)$$

$$= \tilde{\alpha}_{\Sigma}^{-2} \tilde{\alpha}_r \rho \pm 0 \mp \tilde{\beta}_r \vec{u} \cdot (-\omega \tilde{\alpha}^2 \vec{E} + \vec{J}) \quad (3.3.24)$$

$$= \tilde{\alpha}_{\Sigma}^{-2} \tilde{\alpha}_r \rho \pm \omega \tilde{\alpha}^2 \tilde{\beta}_r \vec{u} \cdot \vec{E} \mp \tilde{\beta}_r \vec{u} \cdot \vec{J}. \quad (3.3.25)$$

$$= \tilde{\alpha}_{\Sigma}^{-2} \alpha_r \rho \pm \frac{\omega}{2} \tilde{\alpha}^2 \tilde{\alpha}_r^{-1} \tilde{\beta}_r \vec{u} \cdot [\vec{G}^+ + \vec{G}^-] \mp \tilde{\beta}_r \vec{u} \cdot \vec{J}. \quad (3.3.26)$$

Thus I never use the (no monopoles condition)  $\nabla \cdot \vec{B} = 0$ . Clearly, if  $\vec{G}^{\pm}$  are pure transverse (not just THF, which is built into the definition, but no longitudinal  $E$  fields either), then

$$\nabla \cdot \vec{G}^{\pm}(\omega) = \tilde{\alpha}_{\Sigma}^{-2} \tilde{\alpha}_r \rho \quad (3.3.27)$$

NB:

$$\nabla \cdot \vec{G}^+(\omega) - \nabla \cdot \vec{G}^-(\omega) = \tilde{\alpha}_{\Sigma}^{-2} \tilde{\alpha}_r \rho + \frac{\omega}{2} \tilde{\alpha}^2 \tilde{\alpha}_r^{-1} \tilde{\beta}_r \vec{u} \cdot [\vec{G}^+(\omega) + \vec{G}^-(\omega)] - \tilde{\beta}_r \vec{u} \cdot \vec{J}(\omega) \quad (3.3.28)$$

$$- \tilde{\alpha}_{\Sigma}^{-2} \tilde{\alpha}_r \rho + \frac{\omega}{2} \tilde{\alpha}^2 \tilde{\alpha}_r^{-1} \tilde{\beta}_r \vec{u} \cdot [\vec{G}^+ + \vec{G}^-] - \tilde{\beta}_r \vec{u} \cdot \vec{J} \quad (3.3.29)$$

$$= +\omega \tilde{\alpha}_{\Sigma}^2 \tilde{\alpha}_r^{-1} \tilde{\beta}_r \vec{u} \cdot [\vec{G}^+(\omega) + \vec{G}^-(\omega)] - 2 \tilde{\beta}_r \vec{u} \cdot \vec{J}(\omega). \quad (3.3.30)$$

2. *Circularly polarized*

Handling circularly polarized light is easy using  $\vec{G}^\pm$  fields. Just as for  $E$  and  $H$  fields, you define

$$\vec{E} = \vec{x}E_x \cos(\omega t - \phi) + \vec{y}E_y \cos(\omega t), \quad (3.3.31)$$

$$\vec{H} = -\vec{x}H_x \cos(\omega t) + \vec{y}H_y \cos(\omega t - \phi) \quad (3.3.32)$$

we get

$$\vec{E} \pm \vec{z} \times \vec{H} = \vec{x}[E_x \pm H_y] \cos(\omega t) + \vec{y}[E_y \pm H_x] \cos(\omega t - \phi) \quad (3.3.33)$$

$$\vec{G}^\pm = \vec{x}G_x^\pm \cos(\omega t - \phi) + \vec{y}G_y^\pm \cos(\omega t) \quad (3.3.34)$$

Similar definitions will hold for choices other than the standard “primary  $\vec{E}$ ”  $\vec{G}^\pm$  fields.

3. *Comparison with E and H*

Assume

$$G^+ = Ae^{i(kz-\omega t)}, \quad G^- = 0 \quad (3.3.35)$$

$$E = \frac{A}{2\alpha_r}e^{i(kz-\omega t)}, \quad H = \frac{A}{2\beta_r}e^{i(kz-\omega t)}, \quad (3.3.36)$$

$$\text{so } E/H = \frac{A}{2\alpha_r} \cdot \frac{2\beta_r}{A} = \frac{\beta_r}{\alpha_r} \quad (3.3.37)$$

Compare with the results we get from an  $E$  carrier-based approach

$$E = Be^{i(kz-\omega t)}, \quad (3.3.38)$$

$$\text{to get } H \text{ we use } \mu\partial_t H = \partial_z E \quad (3.3.39)$$

$$\text{so } \mu\partial_t H = ik.Be^{i(kz-\omega t)}, \quad (3.3.40)$$

$$H = \frac{1}{\mu} \frac{ik}{(-\omega)} \cdot Be^{i(kz-\omega t)}, \quad (3.3.41)$$

$$= -\frac{1}{\beta^2} \alpha \beta \cdot Be^{i(kz-\omega t)}, \quad = -\frac{\alpha}{\beta} \cdot Be^{i(kz-\omega t)}, \quad (3.3.42)$$

$$\text{so } E/H = B \cdot \frac{\beta}{\alpha B} = \frac{\beta}{\alpha} \quad (3.3.43)$$

D. Primary  $\vec{H}$ 

If we wanted to swap the roles of  $\vec{H}$  and  $\vec{E}$ , we might instead define the field variables as

$$\vec{G}'^\pm(\omega) = \tilde{\beta}_r(\omega) \vec{H}(\omega) \pm \vec{u} \times \tilde{\alpha}_r(\omega) \vec{E}(\omega), \quad (3.4.1)$$

$$G'^o(\omega) = \vec{u} \cdot \tilde{\alpha}_r(\omega) \vec{E}(\omega), \quad (3.4.2)$$

$$\text{so } \vec{u} \times \tilde{\alpha}_r(\omega) \vec{E}(\omega) = \frac{1}{2} [\vec{G}'^+(\omega) - \vec{G}'^-(\omega)] \quad (3.4.3)$$

$$\text{or } \tilde{\alpha}_r(\omega) \vec{E}(\omega) = \frac{1}{2} \vec{u} \times [\vec{G}'^+(\omega) - \vec{G}'^-(\omega)] + \vec{u} G'^o \quad (3.4.4)$$

$$\text{and } \tilde{\beta}_r(\omega) \vec{H}(\omega) = \frac{1}{2} [\vec{G}'^+(\omega) + \vec{G}'^-(\omega)]. \quad (3.4.5)$$

We would do this so that the magnetic terms not included in the reference medium (i.e. in the definitions of  $\vec{G}'^\pm$ ) would lose the non-transverse correction terms (see later), although terms like those would then appear on the electric terms (and I would define a transverse electric fields (TEF) approximation instead).

A derivation of the wave equations using this form is in section V.



## 1. Divergence

Here we need to calculate  $\vec{B}$  from  $\vec{H}$ , giving us something like  $\vec{B} = \tilde{\beta}_{\Sigma}^{-2} \vec{H}$ ; just as for the Primary  $\vec{E}$  form where we needed to calculate  $\vec{D}$  from  $\vec{E}$ . However, rather than repeat an almost identical calculation, I simply note that  $\nabla \cdot \vec{B} = 0$  (no magnetic monopoles), which will cause the term to vanish anyway.

The divergence calculation is

$$\nabla \cdot \vec{G}'^{\pm}(\omega) = \nabla \cdot \tilde{\beta}_r \vec{H}(\omega) \pm \nabla \cdot (\vec{u} \times \tilde{\alpha}_r \vec{E}(\omega)) \quad (3.4.6)$$

$$= \tilde{\beta}_{\Sigma}^{-2} \tilde{\beta}_r \nabla \cdot \vec{B} \pm \tilde{\alpha}_r \nabla \cdot (\vec{u} \times \vec{E}) \quad (3.4.7)$$

$$= 0 \mp \tilde{\alpha}_r \vec{u} \cdot (\nabla \times \vec{E}) \pm \tilde{\alpha}_r \vec{E} (\nabla \times \vec{u}) \quad (3.4.8)$$

$$= \mp \tilde{\alpha}_r \vec{u} \cdot (\omega \tilde{\beta}^2 \vec{H}) \quad (3.4.9)$$

$$= \mp \tilde{\alpha}_r \tilde{\beta}_r^{-1} \tilde{\beta}^2 \omega (\vec{u} \cdot \tilde{\beta}_r \vec{H}) \quad (3.4.10)$$

$$= \mp \frac{\omega}{2} \tilde{\alpha}_r \tilde{\beta}_r^{-1} \tilde{\beta}^2 \vec{u} \cdot [\vec{G}^+(\omega) + \vec{G}^-(\omega)]. \quad (3.4.11)$$

So no longitudinal magnetic intensity field  $H$  means zero divergence. Clearly, if  $\vec{G}'^{\pm}$  are pure transverse (not just TEF, which is built into the definition, but no longitudinal  $H$  fields either), then the divergence is zero.

E. Primary  $\vec{D}$ 

If we were more interested in  $\vec{D}$  than  $\vec{E}$  (or  $\vec{B}$  than  $\vec{H}$ ), we might instead define variables as

$$\vec{F}'^{\pm}(\omega) = \tilde{\alpha}_r^{-1}(\omega) \vec{D}(\omega) \pm \vec{u} \times \tilde{\beta}_r^{-1}(\omega) \vec{B}(\omega), \quad (3.5.1)$$

$$F^{\circ}(\omega) = \vec{u} \cdot \tilde{\beta}_r^{-1}(\omega) \vec{B}(\omega), \quad (3.5.2)$$

$$\text{so} \quad \vec{D}(\omega) = \frac{\tilde{\alpha}_r(\omega)}{2} [\vec{F}^+(\omega) + \vec{F}^-(\omega)] \quad (3.5.3)$$

$$\text{and} \quad \vec{u} \times \vec{B}(\omega) = \frac{\tilde{\beta}_r(\omega)}{2} [\vec{F}^+(\omega) - \vec{F}^-(\omega)] \quad (3.5.4)$$

$$\text{and} \quad \vec{B}(\omega) = \frac{\tilde{\beta}_r(\omega)}{2} \vec{u} \times [\vec{F}^+(\omega) - \vec{F}^-(\omega)] + \tilde{\beta}_r(\omega) \vec{u} F^{\circ}(\omega). \quad (3.5.5)$$

We would do this so that the displacement field (and magnetic fields) could be easily reconstructed from  $\vec{F}'^{\pm}$ ; and, in analogy to the THF and TEF mentioned above, we would use a transverse magnetic  $B$  field approximation (TBF).

A derivation of the wave equations using this form is in section VI.

## 1. Divergence

$$\nabla \cdot \vec{F}'^{\pm}(\omega) = \nabla \cdot \tilde{\alpha}_r^{-1} \vec{D}(\omega) \pm \nabla \cdot \vec{u} \times \tilde{\beta}_r^{-1} \vec{B}(\omega) \quad (3.5.6)$$

$$= \tilde{\alpha}_r^{-1} \nabla \cdot \vec{D} \pm \tilde{\beta}_r^{-1} \vec{B} \cdot (\nabla \times \vec{u}) \mp \tilde{\beta}_r^{-1} \vec{u} \cdot (\nabla \times \vec{B}) \quad (3.5.7)$$

$$= \tilde{\alpha}_r^{-1} \rho \pm 0 \mp \tilde{\beta}_r^{-1} \tilde{\beta}^2 \vec{u} \cdot (\nabla \times \vec{H}) \quad (3.5.8)$$

$$= \tilde{\alpha}_r^{-1} \rho \mp \tilde{\beta}_r^{-1} \tilde{\beta}^2 \vec{u} \cdot (-\omega \vec{D} + \vec{J}) \quad (3.5.9)$$

$$= \tilde{\alpha}_r^{-1} \rho \pm \frac{\omega}{2} \tilde{\alpha}_r \tilde{\beta}_r^{-1} \tilde{\beta}^2 \vec{u} \cdot [\vec{F}^+(\omega) + \vec{F}^-(\omega)] \mp \tilde{\beta}_r^{-1} \tilde{\beta}^2 \vec{J}(\omega). \quad (3.5.10)$$



Thus I never use the (no monopoles condition)  $\nabla \cdot \vec{B} = 0$ . Clearly, if  $\vec{F}^\pm$  are pure transverse (not just TBF, which is built into the definition, but no longitudinal  $D$  fields either), then the divergence is depends only on the charge density

$$\nabla \cdot \vec{F}^\pm = \tilde{\alpha}_r^{-1} \rho \quad (3.5.11)$$

NB:

$$\nabla \cdot \vec{F}^+ - \nabla \cdot \vec{F}^- = i\omega \tilde{\alpha}_r \tilde{\beta}_r^{-1} \tilde{\beta}^2 \vec{u} \cdot [\vec{F}^+ + \vec{F}^-] - 2\tilde{\beta}_r^{-1} \tilde{\beta}^2 \vec{J} \quad (3.5.12)$$

#### F. Primary $\vec{B}$

If we wanted to swap the roles of  $\vec{B}$  and  $\vec{D}$ , we might instead define variables as

$$\vec{F}'^\pm(\omega) = \tilde{\beta}_r^{-1}(\omega) \vec{B}(\omega) \pm \vec{u} \times \tilde{\alpha}_r^{-1}(\omega) \vec{D}(\omega), \quad (3.6.1)$$

$$F'^o(\omega) = \vec{u} \cdot \tilde{\alpha}_r^{-1}(\omega) \vec{D}(\omega), \quad (3.6.2)$$

$$\text{so } \vec{u} \times \vec{D}(\omega) = \frac{\tilde{\alpha}_r(\omega)}{2} [\vec{F}'^+(\omega) - \vec{F}'^-(\omega)] \quad (3.6.3)$$

$$\text{or } \vec{D}(\omega) = \frac{\tilde{\alpha}_r(\omega)}{2} \vec{u} \times [\vec{F}'^+(\omega) - \vec{F}'^-(\omega)] + \tilde{\alpha}_r(\omega) \vec{u} F'^o(\omega) \quad (3.6.4)$$

$$\text{and } \vec{B}(\omega) = \frac{\tilde{\beta}_r(\omega)}{2} [\vec{F}'^+(\omega) + \vec{F}'^-(\omega)]. \quad (3.6.5)$$

In analogy to the THF and TEF mentioned above, we would use a transverse displacement field approximation (TDF).

A derivation of the wave equations using this form is (not yet) in section VII.

#### 1. Divergence

$$\nabla \cdot \vec{F}'^\pm(\omega) = \nabla \cdot \tilde{\beta}_r^{-1} \vec{B}(\omega) \pm \nabla \cdot (\vec{u} \times \tilde{\alpha}_r^{-1} \vec{D}(\omega)) \quad (3.6.6)$$

$$= 0 \mp \tilde{\alpha}^2 \tilde{\alpha}_r^{-1} \vec{u} \cdot (\nabla \times \vec{E}) \pm \tilde{\alpha}^2 \tilde{\alpha}_r^{-1} \vec{E} (\nabla \times \vec{u}) \quad (3.6.7)$$

$$= \mp \tilde{\alpha}^2 \tilde{\alpha}_r^{-1} \vec{u} \cdot (i\omega \vec{B}) \quad (3.6.8)$$

$$= \mp i\omega \tilde{\alpha}^2 \tilde{\alpha}_r^{-1} \tilde{\beta}_r (\vec{u} \cdot \tilde{\beta}_r^{-1} \vec{B}) \quad (3.6.9)$$

$$= \mp \frac{i\omega}{2} \tilde{\alpha}^2 \tilde{\alpha}_r^{-1} \tilde{\beta}_r \vec{u} \cdot [\vec{F}^+(\omega) + \vec{F}^-(\omega)]. \quad (3.6.10)$$

Clearly, if  $\vec{F}'^\pm$  are pure transverse (not just TDF, which is built into the definition, but no longitudinal  $B$  fields either), then the divergence is zero.

#### G. Comparison to Kolesik et.al

(needs updating with alteration of  $F$ ,  $F'$  definitions at 20050727/8)



Now let us compare the equations Kolesik equations for  $D$  with those for the  $\vec{G}^\pm$  fields. They have

$$\vec{D}(\vec{k})^\pm = \frac{1}{2} \left[ \vec{D}(\vec{k}) \mp \text{sgn}(kz) \frac{k}{\omega(k)} \vec{u} \times \vec{H}(\vec{k}) \right] \quad (3.7.1)$$

$$\Rightarrow \frac{2}{\sqrt{\mu}} (\sqrt{\mu} \vec{D}^\pm) = \vec{D} \mp \frac{1}{c} \vec{u} \times \vec{H} \quad (3.7.2)$$

$$2 (\sqrt{\mu} \vec{D}^\pm) = \sqrt{\mu} \vec{D} \mp \mu \sqrt{\epsilon} \vec{u} \times \vec{H} \quad (3.7.3)$$

$$= \sqrt{\mu} \vec{D} \mp \sqrt{\epsilon} \vec{u} \times (\mu \vec{H}) \quad (3.7.4)$$

$$= \sqrt{\mu} \vec{D} \mp \sqrt{\epsilon} \vec{u} \times \vec{B} \quad (3.7.5)$$

Barring my dropping of the  $\text{sgn}(kz)$  factor, their  $D^\pm$  can clearly be identified with my “primary  $D$ ” formulation of directional fields (see III E). Of course, there are differences, notably that their dispersion is only applied to the magnetic field (either  $H$  or  $B$ ); my “primary  $E$ ” formulation applied  $\epsilon$  dispersion to  $E$  only, and  $\mu$  dispersion to  $H$  only.

They also present an equation for a directional  $H$  which corresponds to the “primary  $B$ ” form ( $F'$ ) as opposed to the  $D$  ( $F$ ) one.

## H. Moving Frame

It is often useful to transform the following wave equation derivations into a moving frame. The frame will be defined by the speed  $c_f = 1/\alpha_f \beta_f$  in analogy to the reference and correction parameters above. A sensible choice of  $c_f$  is probably helpful, e.g.  $c_f = c_r$  or possibly  $c_f = v_g$ , the pulse group velocity.

The (non relativistic)  $z$ -direction frame translation itself is

$$t' = t - z/c_f \quad (3.8.1)$$

$$z' = z \quad (3.8.2)$$

so that

$$\partial_{t'} = \partial_t \quad (3.8.3)$$

$$\partial_{z'} = -c_f^{-1} \partial_t + \partial_z \quad (3.8.4)$$

When doing vector calculations, I will need a vector form of the frame translation. I use

$$\nabla \times \vec{Q} = \nabla' \times \vec{Q} - \alpha_f \beta_f \vec{u} \times \partial_t \vec{Q} \quad (3.8.5)$$

Assuming  $\vec{u}$  is along the  $z$ -direction –

$$\nabla \times \vec{Q} = \hat{x} (\partial_y Q_z - \partial_z Q_y) + \hat{y} (\partial_z Q_x - \partial_x Q_z) + \hat{z} (\partial_x Q_y - \partial_y Q_x) \quad (3.8.6)$$

$$= \hat{x} (\partial_y Q_z - \partial_z' Q_y - \alpha_f \beta_f \partial_t Q_y) + \hat{y} (\partial_z' Q_x - \partial_x Q_z + \alpha_f \beta_f \partial_t Q_x) + \hat{z} (\partial_x Q_y - \partial_y Q_x) \quad (3.8.7)$$

$$= \hat{x} (\partial_y Q_z - \partial_z' Q_y) + \hat{y} (\partial_z' Q_x - \partial_x Q_z) + \hat{z} (\partial_x Q_y - \partial_y Q_x) - \hat{x} \alpha_f \beta_f \partial_t Q_y + \hat{y} \alpha_f \beta_f \partial_t Q_x \quad (3.8.8)$$

$$= \nabla' \times \vec{Q} - \hat{x} \alpha_f \beta_f \partial_t Q_y + \hat{y} \alpha_f \beta_f \partial_t Q_x \quad (3.8.9)$$

I will often find it useful to define the ratio of the reference and frame speeds:

$$\xi = \alpha_f \beta_f / \alpha_r \beta_r \quad (3.8.10)$$

This moving frame is very useful in a space-propagated model since the pulse is held as a function of time, and so it will stay (nearly) centered while propagating forward. Although it has no sensible limit as the frame speed tends to zero, we can recover that case by setting  $\alpha_f \beta_f = 0$  (or  $\xi = 0$ ) and rewriting the primed variables as unprimed ones.



## 1. Frequency domain

In the frequency domain, the space derivative transforms like

$$\partial_{z'} = \imath\omega\tilde{\alpha}_f\tilde{\beta}_f + \partial_z \quad (3.8.11)$$

which seems to leave open the intriguing possibility of a *frequency* dependent (“dispersive”) moving frame by allowing  $\tilde{\alpha}_f$  and  $\tilde{\beta}_f$  a frequency dependence. I could put *all* the linear evolution in the reference, and match it to the frame (so that  $\alpha_f = \alpha_r$  and  $\beta_f = \beta_r$  and  $\xi = 1$ ). This means that the only alterations to the pulse profile would be those due to nonlinear effects.

We might hope to use a dispersive frame in the following way. Imagine a pulse being propagated in a strongly dispersive medium with some nonlinearity; and that the pulse might be expected to broaden by a large factor of (e.g.) twenty during propagation. This would mean we would need a much wider time window in our simulation than suggested by the initial pulse width. In a dispersive frame matched to a dispersive reference medium, our simulated pulse would not broaden *at all* due to dispersion, so we then hope retain a narrow time window throughout the simulation, hence saving computer time. The drawback is that our nonlinear term absorbs the reference dispersion, and this may cause significant pulse broadening of itself, perhaps of similar extent to the dispersion would. Assuming it did give an advantage, at the end of the simulation we would have to take our dispersive-frame pulse and convert back to a non-dispersive frame to see the “lab” pulse profile. Fortunately the computational cost of this conversion is minimal.

## I. Pseudo-Beltrami forms

Weiglhofer and Lakhtakia [11] present a number of quantities related to the EM field using their Beltrami style definitions. If I rearrange the factors of  $\mu$ 's and  $\epsilon$ , I might rewrite these as

$$\text{free field: } \epsilon^{1/2}Q_{\pm} = \epsilon^{1/2}E \pm \imath\mu^{1/2}H \quad (3.9.1)$$

$$\text{coupled field: } \epsilon^{1/2}F_{\pm} = \epsilon^{-1/2}D \pm \imath\mu^{-1/2}B \quad (3.9.2)$$

$$\text{currents: } \mu^{-1/2}W_{\pm} = -\mu^{-1/2}J_m \pm \imath\epsilon^{-1/2}J_e \quad (3.9.3)$$

$$\text{charges: } \mu^{-1/2}w_{\pm} = -\mu^{-1/2}\rho_m \pm \imath\epsilon^{-1/2}\rho_e \quad (3.9.4)$$

Analogously, and since we use  $\vec{u} \times$  the curl- $H$  equation (which has  $D$ ,  $J_e$  on the RHS) –

$$\text{free field: } \vec{G}_{\pm} = \epsilon^{1/2}\vec{E} \pm \mu^{1/2}\vec{u} \times \vec{H} \quad (3.9.5)$$

$$\text{coupled field: } \vec{F}_{\pm} = \epsilon^{-1/2}\vec{u} \times \vec{D} \pm \mu^{-1/2}\vec{B} \quad (3.9.6)$$

$$\text{currents: } \vec{K}_{\pm} = \mu^{1/2}\vec{u} \times \vec{J}_e \pm \epsilon^{1/2}\vec{J}_m \quad (3.9.7)$$

$$\text{charges: } L_{\pm} = \epsilon^{-1/2}\rho_e \pm \mu^{-1/2}\rho_m \quad ?? \quad (3.9.8)$$

It might also be interesting to construct Beltrami-like combinations of the directional  $G^{\pm}$  fields, i.e.

$$\vec{Q}'^{\pm} = \vec{G}^{\pm} + \imath\vec{G}^{\mp}. \quad (3.9.9)$$

## J. Miscellaneous: Force on a moving charge

If I define a velocity  $\vec{v} = v\vec{u}$ , there is a curious similarity between the force on a moving charge and the construction of my  $\vec{G}^+$  field variable. We have –

$$\frac{1}{q}\vec{F} = \vec{E} + \vec{v} \times \vec{B} \quad (3.10.1)$$

$$\frac{\epsilon^{1/2}}{q}\vec{F} = \epsilon^{1/2}\vec{E} + v\epsilon^{1/2}\vec{u} \times \mu\vec{H} \quad (3.10.2)$$

$$= \epsilon^{1/2}\vec{E} + v\epsilon^{1/2}\mu^{1/2}\vec{u} \times \mu^{1/2}\vec{H} \quad (3.10.3)$$

$$= \epsilon^{1/2}\vec{E} + \frac{v}{c}\vec{u} \times \mu^{1/2}\vec{H} \quad (3.10.4)$$



So if  $v = c$ ,

$$\frac{\epsilon^{1/2}}{q} \vec{F} = \epsilon^{1/2} \vec{E} + \vec{u} \times \mu^{1/2} \vec{H} \quad (3.10.5)$$

$$= \vec{G}^+ \quad (3.10.6)$$

This seems to be a remarkable coincidence!



#### IV. FIRST ORDER EVOLUTION EQUATIONS: PRIMARY $\tilde{E}$

Here I derive coupled wave equations for the  $\tilde{G}^\pm$  fields defined in III C. I also discuss how things look in various frames of reference, and in subsection IV B generate first-order differential equations for forward and backward propagating fields. In the following I do not introduce any envelope-carrier decomposition for the field variables; although the introduction of a moving frame plays a related role. Note that the Vector Stationary Frame subsection (IV B) also contains the plane-polarized special case, as well as a transformation to the moving frame as well; and should be considered to contain the authoritative results. Note there are still some sign discrepancies between the vector calculations and the plane polarized ones.

##### A. Plane-polarized stationary frame (Example only)

Note that is better to follow the (authoritative) vector stationary frame derivation (see IV B) instead. However, this is a simpler derivation that gives a nice overview of the approach. It is superceeded by the full vectorized (and moving-frame) derivation(s) below. I begin with the source-free Maxwell's equations –

$$\frac{\partial H_y(t)}{\partial z} = -\frac{\partial}{\partial t}\epsilon * E_x(t), \quad (4.1.1)$$

$$\frac{\partial E_x(t)}{\partial z} = -\frac{\partial}{\partial t}\mu * H_y(t) \quad (4.1.2)$$

Because the convolutions  $\epsilon * E_x$  and  $\mu * H_y$  complicate the analysis, I will Fourier transform into the frequency domain where their linear parts become simple products. Using the definitions in subsection II, and carelessly dropping the tildes off the  $\alpha$  and  $\beta$  parameters, I can proceed with the calculation –

$$\frac{\partial}{\partial z}\tilde{H}_y = \omega\tilde{\epsilon}\tilde{E}_x = \omega\left(\alpha_r^2\tilde{E}_x + \alpha_r\alpha_c * \tilde{E}_x\right), \quad (4.1.3)$$

$$\frac{\partial}{\partial z}\tilde{E}_x = \omega\tilde{\mu}\tilde{H}_y = \omega\left(\beta_r^2\tilde{H}_y + \beta_r\beta_c * \tilde{H}_y\right)^2 \quad (4.1.4)$$

$$\Rightarrow \frac{\partial}{\partial z}\beta_r\tilde{H}_y = \omega\alpha_r\beta_r\left[\alpha_r\tilde{E}_x\right] + \omega\beta_r\alpha_c * \left[\alpha_r\tilde{E}_x\right], \quad (4.1.5)$$

$$\frac{\partial}{\partial z}\alpha_r\tilde{E}_x = \omega\alpha_r\beta_r\left[\beta_r\tilde{H}_y\right] + \omega\alpha_r\beta_c * \left[\beta_r\tilde{H}_y\right] \quad (4.1.6)$$

Adding and subtracting eqns. (4.1.5, 4.1.6), and using the Fourier transform of the definitions for  $G^\pm = \alpha_r E_x \pm \beta_r H_y$ , we get

$$\frac{\partial}{\partial z'}\tilde{G}^\pm = \pm\omega\alpha_r\beta_r\tilde{G}^\pm + \omega\alpha_r\beta_c * \beta_r\tilde{H}_y \pm \omega\beta_r\alpha_c * \alpha_r\tilde{E}_x \quad (4.1.7)$$

$$= \pm\omega\alpha_r\beta_r\tilde{G}^\pm + \omega\frac{\alpha_r}{2}\beta_c * \left(\tilde{G}^+ - \tilde{G}^-\right) \pm \omega\frac{\beta_r}{2}\alpha_c * \left(\tilde{G}^+ + \tilde{G}^-\right) \quad (4.1.8)$$

The “non-magnetic” limit (strictly, no magnetic corrections to the frame), where  $\beta_c = 0$  is

$$\frac{\partial}{\partial z'}\tilde{G}^\pm = \pm\omega\alpha_r\beta_r\tilde{G}^\pm \pm \omega\frac{\beta_r}{2}\alpha_c * \left(\tilde{G}^+ + \tilde{G}^-\right) \quad (4.1.9)$$

The reference-only limit of the above equations (with  $\alpha_1 = 0; \beta_1 = 0$ ) is exactly as you would expect (see eqn.(4.1.10)) –

$$\frac{\partial}{\partial z'}\tilde{G}^\pm = \pm\omega\alpha_r\beta_r G^\pm \quad (4.1.10)$$

Clearly, even if  $\alpha_c \ll \alpha_r$ , both  $G^+$  and  $G^-$  evolve on similar scales, as specified by the leading RHS term. It is interesting to compare Kolesik et.al.’s eqn(KMM 7) (i.e. (1.2.3)) with my eqn (4.1.8) above. Of course their equation has the  $\tilde{G}^\mp$  terms projected out, but otherwise the two are similar in form.



## B. Vector stationary frame (Authoritative)

Here I avoid calculations using the  $x, y$ , and  $z$  components of the field, and retain a fully vector description. Using the definitions in subsection II, I can proceed with the calculation –

$$\begin{aligned}\nabla \times \vec{H}(t) &= +\partial_t \epsilon * \vec{E}(t) + \vec{J}, \\ \nabla \times \vec{E}(t) &= -\partial_t \mu * \vec{H}(t)\end{aligned}\quad (4.2.1)$$

into temporal-frequency space →

$$\begin{aligned}\nabla \times \vec{H}(\omega) &= -i\omega \alpha(\omega)^2 * \vec{E}(\omega) + \vec{J}(\omega), \\ \nabla \times \vec{E}(\omega) &= +i\omega \beta(\omega)^2 * \vec{H}(\omega)\end{aligned}\quad (4.2.2)$$

swap transverse components →

$$\begin{aligned}\vec{u} \times (\nabla \times \vec{H}) &= -i\omega \alpha^2 * (\vec{u} \times \vec{E}) + \vec{u} \times \vec{J}, \\ \nabla \times \vec{E} &= +i\omega \beta^2 * \vec{H}\end{aligned}\quad (4.2.3)$$

premultiply →

$$\begin{aligned}\vec{u} \times (\nabla \times \beta_r \vec{H}) &= -i\omega \beta_r \alpha^2 * (\vec{u} \times \vec{E}) + \vec{u} \times \beta_r \vec{J}, \\ \nabla \times \alpha_r \vec{E} &= +i\omega \alpha_r \beta^2 * \vec{H}\end{aligned}\quad (4.2.4)$$

sum-and-difference →

$$\begin{aligned}\nabla \times \alpha_r \vec{E} \pm \vec{u} \times (\nabla \times \beta_r \vec{H}) &= +i\omega \alpha_r \beta^2 * \vec{H} \mp i\omega \beta_r \alpha^2 * (\vec{u} \times \vec{E}) \\ \pm \vec{u} \times \beta_r \vec{J},\end{aligned}\quad (4.2.5)$$

The vector  $\vec{G}^\pm$  fields (defined in eqn (3.3.1)) are

$$\vec{G}^\pm = \alpha_r \vec{E} + \vec{u} \times \beta_r \vec{H}\quad (4.2.6)$$

This means I need to convert both the second term on the LHS of the sum-and-difference equation above, as well as the RHS. It is most important for the LHS to be simple, because this will define the type of propagation specified by the RHS. To convert the LHS, we need two vector identities, as used in subsection XII to get eqn.(12.0.10). So using

$$\vec{u} \times (\nabla \times \vec{H}) - \nabla (\vec{u} \cdot \vec{H}) = \nabla \times (\vec{u} \times \vec{H})\quad (4.2.7)$$

If I retain the  $\vec{u} \cdot \beta_r \vec{H}$  term , which will be zero for transverse  $H$  fields (THF), then –

$$\nabla \times \alpha_r \vec{E} \pm \vec{u} \times (\nabla \times \beta_r \vec{H}) = +i\omega \alpha_r \beta^2 * \vec{H} \mp i\omega \beta_r \alpha^2 * (\vec{u} \times \vec{E}) \pm \vec{u} \times \beta_r \vec{J}\quad (4.2.8)$$

$$\nabla \times \alpha_r \vec{E} \pm \nabla \times (\vec{u} \times \beta_r \vec{H}) \pm \nabla (\vec{u} \cdot \beta_r \vec{H}) = +i\omega \alpha_r \beta^2 * \vec{H} \mp i\omega \beta_r \alpha^2 * (\vec{u} \times \vec{E}) \pm \vec{u} \times \beta_r \vec{J}\quad (4.2.9)$$

$$\nabla \times [\alpha_r \vec{E} \pm (\vec{u} \times \beta_r \vec{H})] = +i\omega \alpha_r \beta^2 * \vec{H} \mp i\omega \beta_r \alpha^2 * (\vec{u} \times \vec{E}) \mp \nabla (\vec{u} \cdot \beta_r \vec{H}) \pm \vec{u} \times \beta_r \vec{J}\quad (4.2.10)$$

$$\nabla \times \vec{G}^\pm = i\omega \{\alpha_r \beta^2 * \vec{H} \mp \beta_r \alpha^2 * (\vec{u} \times \vec{E})\} \mp \nabla (\vec{u} \cdot \beta_r \vec{H}) \pm \vec{u} \times \beta_r \vec{J}\quad (4.2.11)$$

Now note that eqn.(12.0.12) means that

$$\vec{u} \times [\vec{u} \times \vec{H}] = [\vec{u} \cdot \vec{H}] \vec{u} - \vec{H}\quad (4.2.12)$$

$$\text{and so } \vec{u} \times \vec{G}^\pm = \vec{u} \times \alpha_r \vec{E} \pm \vec{u} \times [\vec{u} \times \beta_r \vec{H}]\quad (4.2.13)$$

$$= \vec{u} \times \alpha_r \vec{E} \mp \beta_r \vec{H} \pm [\vec{u} \cdot \beta_r \vec{H}] \vec{u}\quad (4.2.14)$$

So,

$$\nabla \times \vec{G}^\pm = i\omega \{\alpha_r \beta^2 * [\vec{u} \cdot \vec{H}] \vec{u} - \alpha_r \beta^2 * [\vec{u} \times \vec{H}]\} \mp \beta_r \alpha^2 * (\vec{u} \times \vec{E}) \mp \nabla (\vec{u} \cdot \beta_r \vec{H}) \pm \vec{u} \times \beta_r \vec{J}\quad (4.2.15)$$

$$= \mp i\omega \{\beta_r \alpha^2 * (\vec{u} \times \vec{E}) \pm \alpha_r \beta^2 * [\vec{u} \times \vec{H}]\} \pm i\omega \alpha_r \beta^2 * [\vec{u} \cdot \vec{H}] \vec{u} \mp \nabla (\vec{u} \cdot \beta_r \vec{H}) \pm \vec{u} \times \beta_r \vec{J}\quad (4.2.16)$$



I now separate the interaction parts (depending on  $\alpha_c, \beta_c$ ) from the vacuum-like parts (depending on  $\alpha_r, \beta_r$ ), and then substitute (as far as possible) expressions containing  $G$  rather than  $E$  or  $H$ , by referring to eqns (3.3.3) and (3.3.4). Hence

$$\begin{aligned}\nabla \times \vec{G}^\pm &= \mp i\omega\alpha_r\beta_r \left\{ \vec{u} \times \alpha_r \vec{E} \pm \vec{u} \times [\vec{u} \times \beta_r \vec{H}] \right\} + i\omega\alpha_r\beta_r [\vec{u} \cdot \beta_r \vec{H}] \vec{u} \mp \nabla (\vec{u} \cdot \beta_r \vec{H}) \\ &\quad \mp i\omega\beta_r\alpha_c * (\vec{u} \times \alpha_r \vec{E}) - i\omega\alpha_r\beta_c * (\vec{u} \times [\vec{u} \times \beta_r \vec{H}]) + i\omega\alpha_r\beta_c * [\vec{u} \cdot \beta_r \vec{H}] \vec{u} \pm \vec{u} \times \beta_r \vec{J}\end{aligned}\quad (4.2.17)$$

$$\begin{aligned}&= \mp i\omega\alpha_r\beta_r \vec{u} \times \vec{G}^\pm + i\omega\alpha_r\beta_r \vec{u} G^\circ \mp \nabla G^\circ \\ &\quad \mp \frac{i\omega\beta_r}{2} \alpha_c * (\vec{u} \times [\vec{G}^+ + \vec{G}^-]) - \frac{i\omega\alpha_r}{2} \beta_c * (\vec{u} \times [\vec{G}^+ - \vec{G}^-]) + i\omega\alpha_r\beta_c * (\vec{u} G^\circ) \pm \vec{u} \times \beta_r \vec{J}\end{aligned}\quad (4.2.18)$$

Here the transverse (THF) and longitudinal parts decouple, since  $\vec{G}^\pm$  is guaranteed magnetically transverse, and  $\vec{u} G^\circ$  is magnetically longitudinal. The two decoupled equations are –

$$\nabla \times \vec{G}^\pm = \mp i\omega\alpha_r\beta_r \vec{u} \times \vec{G}^\pm \mp \frac{i\omega\beta_r}{2} \alpha_c * (\vec{u} \times [\vec{G}^+ + \vec{G}^-]) - \frac{i\omega\alpha_r}{2} \beta_c * (\vec{u} \times [\vec{G}^+ - \vec{G}^-]) \pm \vec{u} \times \beta_r \vec{J} \quad (4.2.19)$$

$$\pm \nabla G^\circ = +i\omega\alpha_r\beta_r \vec{u} G^\circ + i\omega\alpha_r\beta_c * (\vec{u} G^\circ). \quad (4.2.20)$$

We see from the following IV B 1 that these two equations are sufficient to describe the fields as long as there is no longitudinal  $\vec{E}$  component (i.e.  $\vec{G}^\pm$  is transverse) or longitudinal  $\vec{J}$  (also with reference to the divergence calculation in III C 1).

### 1. Longitudinal Equation

Taking the  $\vec{u} \times$  of the  $\nabla \times \vec{H}$  equation removed a part of the field dynamics lying in the direction of the propagation vector  $\vec{u}$ . This part also include the response to longitudinal currents. To rectify this omission, we take the dot product with  $\vec{u}$ . Using the standard vector identity eqn.(12.0.13):

$$\nabla \cdot (\vec{A} \times \vec{B}) = \vec{A} \cdot (\nabla \times \vec{B}) - \vec{B} \cdot (\nabla \times \vec{A}), \quad (4.2.21)$$

and using  $\nabla \times \vec{u} = 0$ ; we get

$$\vec{u} \cdot (\nabla \times \vec{H}) = -i\omega\alpha^2 * (\vec{u} \cdot \vec{E}) + \vec{u} \cdot \vec{J} \quad (4.2.22)$$

$$\text{vector identity: } \nabla \cdot (\vec{u} \times \vec{H}) = -i\omega\alpha^2 * (\vec{u} \cdot \vec{E}) + \vec{u} \cdot \vec{J} \quad (4.2.23)$$

$$\text{premultiply: } \alpha_r \nabla \cdot (\vec{u} \times \beta_r \vec{H}) = -i\omega\beta_r\alpha^2 * (\vec{u} \cdot \alpha_r \vec{E}) + \alpha_r\beta_r \vec{u} \cdot \vec{J} \quad (4.2.24)$$

$$\alpha_r \nabla \cdot (\vec{G}^+ - \vec{G}^-) = -i\omega\beta_r\alpha^2 * (\vec{u} \cdot [\vec{G}^+ + \vec{G}^-]) + 2\alpha_r\beta_r \vec{u} \cdot \vec{J} \quad (4.2.25)$$

This is the same as the source-free divergence-difference equation seen in III C 1; where we see that both parts of the LHS are zero if  $\vec{u} \cdot (\vec{G}^+ + \vec{G}^-) = 0$  and  $\vec{u} \cdot \vec{J} = 0$ .

### 2. Simple Cases

In the non-magnetic ( $\beta_c = 0$ ) case,

$$\nabla \times \vec{G}^\pm = \mp i\omega\alpha_r\beta_r \vec{u} \times \vec{G}^\pm + i\omega\alpha_r\beta_r \vec{G}^\circ \mp \frac{i\omega\beta_r}{2} \alpha_c * (\vec{u} \times [\vec{G}^+ + \vec{G}^-]) \pm \vec{u} \times \beta_r \vec{J} \quad (4.2.26)$$

$$\pm \nabla G^\circ = +i\omega\alpha_r\beta_r \vec{u} G^\circ \quad (4.2.27)$$

In the time domain, eqn.(4.2.26) is

$$\nabla \times \vec{G}^\pm = \pm \partial_t \left[ (\alpha_r * \beta_r) * (\vec{u} \times \vec{G}^\pm) \right] \pm \partial_t \left[ \left( \frac{\alpha_c * \beta_r}{2} \right) * (\vec{u} \times [\vec{G}^+ + \vec{G}^-]) \right] \pm \vec{u} \times \beta_r * \vec{J}. \quad (4.2.28)$$

**Plane Polarized & Magnetic ( $G_x^\pm \sim E_x \pm H_y$ ):**

The  $\hat{y}$  components of the LHS and RHS contain the  $G_x$  values – as a result of the curl and cross-products respectively. Compare the previous stand-alone calculation in eqn.(4.1.8) to

$$\partial_z G_x^\pm = \mp i\omega\alpha_r\beta_r G_x^\pm \mp \frac{i\omega\beta_r}{2}\alpha_c * [G_x^+ + G_x^-] - \frac{i\omega\alpha_r}{2}\beta_c * [G_x^+ - G_x^-] \quad (4.2.29)$$

This contains a single sign discrepancy compared to eqn (4.1.8); the RHS has the opposite sign.

NB: recheck curl/cross-product component signs? (checked OK 20041028)

**Vector Moving Frame, Transverse, Magnetic:**

It is worth transforming to the moving frame at this end point, rather than at the beginning as in subsection IV D. Since eqn.(3.8.5) tells us that  $\nabla' \vec{Q} = \nabla' \vec{Q} - \alpha_f \beta_f \vec{u} \times \partial'_t \vec{Q}$ , with  $\xi = \alpha_f \beta_f / \alpha_r \beta_r$ , we get

$$\nabla' \times \vec{G}^\pm + i\omega\alpha_f\beta_f \vec{u} \times \vec{G}^\pm = \mp i\omega\alpha_r\beta_r \vec{u} \times \vec{G}^\pm \mp \frac{i\omega\beta_r}{2}\alpha_c * (\vec{u} \times [\vec{G}^+ + \vec{G}^-]) - \frac{i\omega\alpha_r}{2}\beta_c * (\vec{u} \times [\vec{G}^+ - \vec{G}^-]) \quad (4.2.30)$$

$$\begin{aligned} \nabla' \times \vec{G}^\pm &= \mp i\omega\alpha_r\beta_r (1 \mp \xi) \vec{u} \times \vec{G}^\pm \\ &\mp \frac{i\omega\beta_r}{2}\alpha_c * (\vec{u} \times [\vec{G}^+ + \vec{G}^-]) - \frac{i\omega\alpha_r}{2}\beta_c * (\vec{u} \times [\vec{G}^+ - \vec{G}^-]) \end{aligned} \quad (4.2.31)$$

This agrees with the standalone vector moving frame calculation in the transverse case (see subsection IV D); which culminates in eqn (4.4.22). I use the transverse case (not the general) to avoid the moving-frame altered  $\nabla$  that occurs in one of the terms.

**Plane Polarized Moving Frame, Magnetic:**

This is just a special case of the above vector moving frame equation –

$$\partial_{z'} G_x^\pm = \mp i\omega\alpha_r\beta_r (1 \mp \xi) G_x^\pm \mp \frac{i\omega\beta_r}{2}\alpha_c * [G_x^+ + G_x^-] - \frac{i\omega\alpha_r}{2}\beta_c * [G_x^+ - G_x^-] \quad (4.2.32)$$

This has sign conflicts with the stand-alone calculation in subsection IV C which culminates in eqn (4.3.14).

**Beltrami-like form:**

This might be written

$$\{(\nabla \times) \pm i\omega\alpha_r\beta_r (\vec{u} \times)\} \vec{G}^\pm = \vec{W}^\pm, \quad (4.2.33)$$

where  $\vec{W}^\pm$  is a “source” term, but note that actually it will be some complicated function of  $\vec{G}^\pm$ .

*3. Comments*

Note the  $\omega\alpha_r\beta_r = \omega/c_f$  prefactors on the RHS's – these trivially convert  $(\omega/c_r \rightarrow \omega/\omega k_r^{-1} \rightarrow k_r)$  to a wavevector prefactor  $k_r$ , as would be expected for this kind of spatially propagating description.

Inspection of the equations shows that even if (e.g.)  $G^+$  starts identically equal to zero, that it will pick up some value due to the interaction of the  $G^-$  components with the non-vacuum part of the permittivity. However, so as long as  $\alpha_c \ll \alpha_r$ , the transfer will be slow, and because of the leading  $2i\omega$  term,  $G^+$  will rapidly oscillate. As a result, we can assume (in an appropriate limit) that any  $G^+$  contribution to the evolution of  $G^-$  will spatially average to zero in a kind of rotating wave approximation (RWA), although we should properly define criteria for “slow transfer” and “RWA”

One approach that would avoid throwing away the backward propagating terms would be to describe the  $G^+$  field with the reverse frame transformation. Whilst this would result in some inconvenience, when simulating numerically it could be useful in that fields propagating in the unimportant direction could be thrown away only once they had ceased overlapping with the range covered by the pulse travelling in the important direction.

**C. Plane-polarized moving frame (Example only)**

NOTE: In this example calculation, the convolutions required for the nonlinearity are not explicitly included in the notation. This does not affect the result, as long as they are reinserted correctly in the final equations.



Here I use the (non relativistic) frame translation defined by eqn.(3.8.4) in subsection III H. Starting with the source-free Maxwell's equations, we have –

$$\frac{\partial H_y(t)}{\partial z} = -\frac{\partial}{\partial t}\epsilon * E_x(t), \quad (4.3.1)$$

$$\frac{\partial E_x(t)}{\partial z} = -\frac{\partial}{\partial t}\mu * H_y(t); \quad (4.3.2)$$

$$\implies \left[ \frac{\partial}{\partial z'} + \alpha_f \beta_f \frac{\partial}{\partial t'} \right] H_y = -\frac{\partial}{\partial t'} \epsilon * E_x, \quad (4.3.3)$$

$$\left[ \frac{\partial}{\partial z'} + \alpha_f \beta_f \frac{\partial}{\partial t'} \right] E_x = -\frac{\partial}{\partial t'} \mu * H_y; \quad (4.3.4)$$

$$\implies \frac{\partial}{\partial z'} H_y = -\frac{\partial}{\partial t'} [\epsilon * E_x + \alpha_f \beta_f H_y], \quad (4.3.5)$$

$$\frac{\partial}{\partial z'} E_x = -\frac{\partial}{\partial t'} [\mu * H_y + \alpha_f \beta_f E_x] \quad (4.3.6)$$

Now, because the convolutions  $\epsilon * E_x$  and  $\mu * H_y$  complicate the analysis, I will Fourier transform into the frequency domain where the linear response becomes a simple product. Using the definitions in subsection II, I can proceed with the calculation –

$$\frac{\partial}{\partial z'} \tilde{H}_y(\omega) = i\omega \left[ \alpha(\omega)^2 \tilde{E}_x(\omega) + \alpha_f(\omega) \beta_f(\omega) \tilde{H}_y(\omega) \right], \quad (4.3.7)$$

$$\frac{\partial}{\partial z'} \tilde{E}_x(\omega) = i\omega \left[ \beta^2 \tilde{H}_y(\omega) + \alpha_f(\omega) \beta_f(\omega) \tilde{E}_x(\omega) \right] \quad (4.3.8)$$

$$\implies \frac{\partial}{\partial z'} \tilde{H}_y = i\omega \left[ (\alpha_r^2 + \alpha_r \alpha_c)^2 \tilde{E}_x + \alpha_f \beta_f \tilde{H}_y \right], \quad (4.3.9)$$

$$\frac{\partial}{\partial z'} \tilde{E}_x = i\omega \left[ (\beta_r^2 + \beta_r \beta_c)^2 \tilde{H}_y + \alpha_f \beta_f \tilde{E}_x \right] \quad (4.3.10)$$

$$\implies \frac{\partial}{\partial z'} \beta_r \tilde{H}_y = i\omega \alpha_r \beta_r \left[ \alpha_r \tilde{E}_x + \xi \beta_r \tilde{H}_y \right] + i\omega \beta_r \alpha_c \cdot \alpha_r \tilde{E}_x, \quad (4.3.11)$$

$$\frac{\partial}{\partial z'} \alpha_r \tilde{E}_x = i\omega \alpha_r \beta_r \left[ \beta_r \tilde{H}_y + \xi \alpha_r \tilde{E}_x \right] + i\omega \alpha_r \beta_c \cdot \beta_r \tilde{H}_y \quad (4.3.12)$$

Adding and subtracting eqns. (4.3.11, 4.3.12), and using the Fourier transform of the definitions for  $G^\pm = \alpha_r E_x \pm \beta_r H_y$ , we get

$$\frac{\partial}{\partial z'} \tilde{G}^\pm = \pm (1 \pm \xi) i\omega \alpha_r \beta_r \tilde{G}^\pm + i\omega \alpha_r \beta_c \cdot \beta_r \tilde{H}_y \pm i\omega \beta_r \alpha_c \cdot \alpha_r \tilde{E}_x \quad (4.3.13)$$

$$= \pm (1 \pm \xi) i\omega \alpha_r \beta_r \tilde{G}^\pm + i\omega \frac{\alpha_r \beta_c}{2} \left( \tilde{G}^+ - \tilde{G}^- \right) \pm i\omega \frac{\beta_r \alpha_c}{2} \left( \tilde{G}^+ + \tilde{G}^- \right) \quad (4.3.14)$$

The “non-magnetic” limit, where  $\beta_1 = 0$  is

$$\frac{\partial}{\partial z'} \tilde{G}^\pm = \pm (1 \pm \xi) \frac{i\omega}{c_f} \tilde{G}^\pm \pm i\omega \frac{\beta_r \alpha_c}{2} \left( \tilde{G}^+ + \tilde{G}^- \right) \quad (4.3.15)$$

The reference limit of the above equations (with  $\alpha_1 = 0; \beta_1 = 0$ ) when the frame is matched to the reference  $c_f = c_r$  is exactly as you would expect, given that the new frame is moving in the direction of the  $G^-$  component, and since the pulse does not evolve (relative to its centre) in a vacuum,  $\partial_{z'} G^-$  is zero (see eqn.(4.3.17)). Conversely, the  $G^+$  component is moving the opposite way to our frame at  $c$ , thus relative to the frame it moves at  $2c$ , hence the derivative is double that in a stationary frame (see eqn.(4.3.16)).

$$\frac{\partial}{\partial z'} \tilde{G}^+ = \pm 2 \frac{i\omega}{c_f} G^+ \quad (4.3.16)$$

$$\frac{\partial}{\partial z'} \tilde{G}^- = 0 \quad (4.3.17)$$



## D. Vector moving frame (Example only)

NOTE: In this example calculation, the nonlinear convolutions are not explicitly included in the notation. This does not affect the result, as long as they are reinserted correctly in the final equations.

Note that is better to follow the (authoritative) vector stationary frame derivation (see IV B) instead; where the frame translation is applied to the wave equation. I use the (non relativistic) frame translation defined by eqn.(3.8.5) in subsection III H; and using the definitions in section II, I can proceed with the calculation –

$$\begin{aligned}\nabla \times \vec{H}(t) &= +\partial_t \epsilon * \vec{E}(t), \\ \nabla \times \vec{E}(t) &= -\partial_t \mu * \vec{H}(t)\end{aligned}\quad (4.4.1)$$

into moving frame →  $\begin{aligned}\nabla' \times \vec{H} &= +\partial_{t'} \epsilon * \vec{E} - \alpha_f \beta_f \vec{u} \times \partial_{t'} \vec{H}, \\ \nabla' \times \vec{E} &= -\partial_{t'} \mu * \vec{H} - \alpha_f \beta_f \vec{u} \times \partial_{t'} \vec{E}\end{aligned}\quad (4.4.2)$

into temporal-frequency space →  $\begin{aligned}\nabla' \times \vec{H}(\omega) &= -i\omega \left( \alpha(\omega)^2 \vec{E}(\omega) - \alpha_f(\omega) \beta_f(\omega) \vec{u} \times \vec{H}(\omega) \right), \\ \nabla' \times \vec{E} &= +i\omega \left( \beta(\omega)^2 \vec{H}(\omega) + \alpha_f(\omega) \beta_f(\omega) \vec{u} \times \vec{E}(\omega) \right)\end{aligned}\quad (4.4.3)$

swap transverse →  $\begin{aligned}\vec{u} \times (\nabla' \times \vec{H}) &= -i\omega \left( \alpha^2 \vec{u} \times \vec{E} - \alpha_f \beta_f \vec{u} \times [\vec{u} \times \vec{H}] \right), \\ \nabla' \times \vec{E} &= +i\omega \left( \beta^2 \vec{H} + \alpha_f \beta_f \vec{u} \times \vec{E} \right)\end{aligned}\quad (4.4.4)$

premultiply →  $\begin{aligned}\vec{u} \times (\nabla' \times \beta_r \vec{H}) &= -i\omega \left( \beta_r \alpha^2 \vec{u} \times \vec{E} - \alpha_f \beta_f \beta_r \vec{u} \times [\vec{u} \times \vec{H}] \right), \\ \nabla' \times \alpha_r \vec{E} &= +i\omega \left( \alpha_r \beta^2 \vec{H} + \alpha_f \beta_f \alpha_r \vec{u} \times \vec{E} \right)\end{aligned}\quad (4.4.5)$

sum & diff →  $\begin{aligned}\nabla' \times \alpha_r \vec{E} \pm \vec{u} \times (\nabla' \times \beta_r \vec{H}) &= +i\omega \left( \alpha_r \beta^2 \vec{H} + \alpha_f \beta_f \alpha_r \vec{u} \times \vec{E} \right) \\ &\mp i\omega \left( \beta_r \alpha^2 \vec{u} \times \vec{E} - \alpha_f \beta_f \beta_r \vec{u} \times [\vec{u} \times \vec{H}] \right)\end{aligned}\quad (4.4.6)$

collect terms →  $\begin{aligned}&= \mp i\omega \left( \alpha^2 \beta_r \mp \alpha_f \beta_f \alpha_r \right) \vec{u} \times \vec{E} \\ &+ i\omega \left( \alpha_r \beta^2 \vec{H} \pm \alpha_f \beta_f \beta_r \vec{u} \times [\vec{u} \times \vec{H}] \right),\end{aligned}\quad (4.4.7)$

The vector  $\vec{G}^\pm$  fields (defined in eqn (3.3.1)) are

$$\vec{G}^\pm = \alpha_r \vec{E} + \vec{u} \times \beta_r \vec{H} \quad (4.4.8)$$

This means I need to convert both the second term on the LHS of the sum-and-difference equation above, as well as the RHS. It is most important for the LHS to be simple, because this will define the type of propagation specified by the RHS. To convert the LHS, we need two vector identities, as used in subsection XII to get eqn.(12.0.10). So using

$$\vec{u} \times (\nabla \times \vec{H}) - \nabla (\vec{u} \cdot \vec{H}) = \nabla \times (\vec{u} \times \vec{H}) \quad (4.4.9)$$

WARNING! These are moving frame  $\nabla'$ 's arriving from the vector identities!!

If I retain the  $\vec{u} \cdot \beta_r \vec{E}$  term, which would be zero for strictly transverse  $E$  fields (TEF), then –

$$\begin{aligned}\nabla' \times \alpha_r \vec{E} \pm \vec{u} \times (\nabla' \times \beta_r \vec{H}) &= \mp i\omega \left( \alpha^2 \beta_r \mp \alpha_f \beta_f \alpha_r \right) \vec{u} \times \vec{E} \\ &+ i\omega \left( \alpha_r \beta^2 \vec{H} \pm \alpha_f \beta_f \beta_r \vec{u} \times [\vec{u} \times \vec{H}] \right)\end{aligned}\quad (4.4.10)$$

$$\begin{aligned}\rightarrow \nabla' \times \alpha_r \vec{E} \pm \nabla' \times (\vec{u} \times \beta_r \vec{H}) \pm \nabla' (\vec{u} \cdot \beta_r \vec{H}) &= \mp i\omega \left( \alpha^2 \beta_r \mp \alpha_f \beta_f \alpha_r \right) \vec{u} \times \vec{E} \\ &+ i\omega \left( \alpha_r \beta^2 \vec{H} \pm \alpha_f \beta_f \beta_r \vec{u} \times [\vec{u} \times \vec{H}] \right)\end{aligned}\quad (4.4.11)$$

$$\begin{aligned}(C) \rightarrow \nabla' \times [\alpha_r \vec{E} \pm (\vec{u} \times \beta_r \vec{H})] &= \mp i\omega \left( \alpha^2 \beta_r \mp \alpha_f \beta_f \alpha_r \right) \vec{u} \times \vec{E} \\ &+ i\omega \left( \alpha_r \beta^2 \vec{H} \pm \alpha_f \beta_f \beta_r \vec{u} \times [\vec{u} \times \vec{H}] \right) \\ &\mp \nabla' (\vec{u} \cdot \beta_r \vec{H})\end{aligned}\quad (4.4.12)$$



Now note that eqn.(12.0.12) means that

$$\vec{u} \times [\vec{u} \times \vec{H}] = [\vec{u} \cdot \vec{H}] \vec{u} - \vec{H} \quad (4.4.13)$$

$$\text{and so } \vec{u} \times \vec{G}^\pm = \vec{u} \times \alpha_r \vec{E} \pm \vec{u} \times [\vec{u} \times \beta_r \vec{H}] \quad (4.4.14)$$

$$= \vec{u} \times \alpha_r \vec{E} \mp \beta_r \vec{H} \pm [\vec{u} \cdot \beta_r \vec{H}] \vec{u} \quad (4.4.15)$$

So,

$$\nabla' \times \vec{G}^\pm = \mp \omega (\alpha^2 \beta_r \mp \alpha_f \beta_f \alpha_r) \vec{u} \times \vec{E} + \omega (\alpha_r \beta^2 \vec{H} \pm \alpha_f \beta_f \beta_r \vec{u} \times [\vec{u} \times \vec{H}]) \mp \nabla' (\vec{u} \cdot \beta_r \vec{H}) \quad (4.4.16)$$

$$= \mp \omega (\alpha^2 \beta_r \mp \alpha_f \beta_f \alpha_r) \vec{u} \times \vec{E} + \omega (\alpha_r \beta^2 [\vec{u} \cdot \vec{H}] \vec{u} - \alpha_r \beta^2 \vec{u} \times [\vec{u} \times \vec{H}] \pm \alpha_f \beta_f \beta_r \vec{u} \times [\vec{u} \times \vec{H}]) \mp \nabla' (\vec{u} \cdot \beta_r \vec{H}) \quad (4.4.17)$$

$$= \mp \omega (\alpha^2 \beta_r \mp \alpha_f \beta_f \alpha_r) \vec{u} \times \vec{E} - \omega (\alpha_r \beta^2 \mp \alpha_f \beta_f \beta_r) \vec{u} \times [\vec{u} \times \vec{H}] + \omega \alpha_r \beta^2 [\vec{u} \cdot \vec{H}] \vec{u} \mp \nabla' (\vec{u} \cdot \beta_r \vec{H}) \quad (4.4.18)$$

I now separate the interaction parts (depending on  $\alpha_c, \beta_c$ ) from the reference parts (depending on  $\alpha_r, \beta_r$ ), and then substitute (as far as possible) expressions containing  $G$  rather than  $E$  or  $H$ , by referring to eqns (3.3.3) and (3.3.4). Hence

$$\begin{aligned} \nabla \times \vec{G}^\pm &= \mp \omega \alpha_r \beta_r (1 \mp \xi) \vec{u} \times \alpha_r \vec{E} - \omega \alpha_r \beta_r (1 \mp \xi) \vec{u} \times [\vec{u} \times \beta_r \vec{H}] + \omega \alpha_r \beta_r [\vec{u} \cdot \beta_r \vec{H}] \vec{u} \mp \nabla' (\vec{u} \cdot \beta_r \vec{H}) \\ &\quad \mp \omega \alpha_c \beta_r \vec{u} \times \alpha_r \vec{E} - \omega \alpha_r \beta_c \vec{u} \times [\vec{u} \times \beta_r \vec{H}] + \omega \alpha_r \beta_c [\vec{u} \cdot \beta_r \vec{H}] \vec{u} \end{aligned} \quad (4.4.19)$$

$$\begin{aligned} &= \mp \omega \alpha_r \beta_r (1 \mp \xi) \{ \vec{u} \times \alpha_r \vec{E} \pm \vec{u} \times [\vec{u} \times \beta_r \vec{H}] \} + \omega \alpha_r \beta_r [\vec{u} \cdot \beta_r \vec{H}] \vec{u} \mp \nabla' (\vec{u} \cdot \beta_r \vec{H}) \\ &\quad \mp \omega \alpha_c \beta_r \vec{u} \times \alpha_r \vec{E} - \omega \alpha_r \beta_c \vec{u} \times [\vec{u} \times \beta_r \vec{H}] + \omega \alpha_r \beta_c [\vec{u} \cdot \beta_r \vec{H}] \vec{u} \end{aligned} \quad (4.4.20)$$

$$\begin{aligned} &= \mp \omega \alpha_r \beta_r (1 \mp \xi) \vec{u} \times \vec{G}^\pm + \omega \alpha_r \beta_r [\vec{u} \cdot \beta_r \vec{H}] \vec{u} \mp \nabla' (\vec{u} \cdot \beta_r \vec{H}) \\ &\quad \mp \omega \alpha_c \beta_r \vec{u} \times \alpha_r \vec{E} - \omega \alpha_r \beta_c \vec{u} \times [\vec{u} \times \beta_r \vec{H}] + \omega \alpha_r \beta_c [\vec{u} \cdot \beta_r \vec{H}] \vec{u} \end{aligned} \quad (4.4.21)$$

Note that the non-transverse corrections depend only on the *Magnetic* field  $\vec{H}$ . I assume this is a curiosities of the chosen  $G$  definition, as (I think) choosing  $G^\pm \sim u \times E \pm H$  would produce non-transverse corrections depending on the electric field. As yet I do not have a simple vectorised form for calculating  $H$  purely in terms of  $G$ 's without these non-transverse corrections occurring.

So using  $\vec{E} = (\vec{G}^+ + \vec{G}^-)/2\alpha_r$ ;  $\vec{u} \times \vec{H} = (\vec{G}^+ - \vec{G}^-)/2\beta_r$ ;  $\vec{H} = \vec{u} \times (\vec{u} \times \vec{H}) - (\vec{u} \cdot \vec{H})\vec{u}$

$$\begin{aligned} \nabla' \times \vec{G}^\pm &= \mp \omega \alpha_r \beta_r (1 \mp \xi) \vec{u} \times \vec{G}^\pm + \omega \alpha_r \beta_r [\vec{u} \cdot \beta_r \vec{H}] \vec{u} \mp \nabla' (\vec{u} \cdot \beta_r \vec{H}) \\ &\quad \mp \frac{\omega \alpha_c \beta_r}{2} \vec{u} \times [\vec{G}^+ + \vec{G}^-] - \frac{\omega \alpha_r \beta_c}{2} [\vec{u} \times (\vec{G}^+ - \vec{G}^-)] + \omega \alpha_r \beta_c [\vec{u} \cdot \beta_r \vec{H}] \vec{u} \end{aligned} \quad (4.4.22)$$

$$\text{non-magnetic} \rightarrow = \mp \omega \alpha_r \beta_r (1 \mp \xi) \vec{u} \times \vec{G}^\pm \mp \frac{\omega \alpha_c \beta_r}{2} \vec{u} \times [\vec{G}^+ + \vec{G}^-] + \omega \alpha_r \beta_r [\vec{u} \cdot \beta_r \vec{H}] \vec{u} \mp \nabla' (\vec{u} \cdot \beta_r \vec{H}) \quad (4.4.23)$$

$$\text{transverse} \rightarrow = \mp \omega \alpha_r \beta_r (1 \mp \xi) \vec{u} \times \vec{G}^\pm \mp \frac{\omega \alpha_c \beta_r}{2} \vec{u} \times [\vec{G}^+ + \vec{G}^-] \quad (4.4.24)$$



## E. Simulations: a step by step guide

### 1. The medium

Consider a plane polarized field propagating through a non-magnetic medium. I choose a stationary frame, and a dispersionless linear reference (so  $\alpha_r, \beta_r$  are constants). The permeability of the medium is the vacuum value  $\mu_0$ , and the permittivity has three contributions: a vacuum (reference)  $\epsilon_0$ , a linear dispersion  $\epsilon_D(t)$ , and an instantaneous nonlinearity  $\epsilon_{NL}$ . The medium properties are therefore broken down in the following fashion –

$$\tilde{\epsilon} = \epsilon_0 + \tilde{\epsilon}_c^D(\omega) + \epsilon_c^{NL} \quad (4.5.1)$$

$$\tilde{\epsilon} = \tilde{\alpha}_r^2 + \tilde{\epsilon}_c^D \tilde{\alpha}_r + \tilde{\epsilon}_c^{NL} \tilde{\alpha}_r \quad (4.5.2)$$

$$\text{so} \quad \tilde{\epsilon}_c^D = \tilde{\epsilon}_c^D(\omega)/\tilde{\alpha}_r \quad (4.5.3)$$

$$\tilde{\epsilon}_c^{NL} = \tilde{\epsilon}_c^{NL}(\omega)/\tilde{\alpha}_r. \quad (4.5.4)$$

We can see here that choosing a dispersive reference medium would modify both the dispersive correction and the nonlinear term. In the nonlinear case, this adds a time dependence which did not previously exist, which would increase the computational effort needed if the nonlinearity  $\tilde{\epsilon}_c^{NL}$  was instantaneous.

### 2. The full wave equation

The first order evolution equation, using the parameter definitions at eqns.(2.1.6, 2.1.8)), is

$$\partial_z G_x^\pm = \mp i\omega \tilde{\alpha}_r \tilde{\beta}_r (1 \mp \xi) G_x^\pm \mp \frac{i\omega \tilde{\alpha}_c \tilde{\beta}_r}{2} [G_x^+ + G_x^-] \quad (4.5.5)$$

$$= \mp i\omega \tilde{\alpha}_r \tilde{\beta}_r (1 \mp \xi) G_x^\pm \mp \frac{i\omega \tilde{\alpha}_c^D \tilde{\beta}_r}{2} [G_x^+ + G_x^-] \mp \frac{i\omega \tilde{\beta}_r}{2} \tilde{\alpha}_c^{NL} * [G_x^+ + G_x^-] \quad (4.5.6)$$

This has three terms – a reference carrier-oscillation-like term ( $\sim \tilde{\alpha}_r$ ), a linear dispersion term ( $\sim \tilde{\alpha}_c^D$ ), and a nonlinear polarization term ( $\sim \tilde{\alpha}_c^{NL}$ ). The straightforward way of solving this using a split step method, where each contribution is considered to be relatively weak, hence we solve the above in three steps. Each one integrates forward a distance  $\delta z$ , but since we do one after the other (rather than doing them simultaneously), the procedure is only accurate to first order – but for small enough  $\delta z$ , the error can be assured negligible.

### 3. The reference evolution

The reference term just applies a complex rotation to the field in frequency space, which can be calculated exactly

$$G_{x1}^\pm = G_x^\pm(z) \times \exp [\mp i\omega \tilde{\alpha}_r \tilde{\beta}_r (1 \mp \xi) \cdot \delta z]. \quad (4.5.7)$$

This will have no effect if we choose the frame velocity to be the same as the phase velocity of the reference medium. This might seem to differ from the usual  $E$  field approaches, where choosing the frame velocity to match the group velocity gives the best cancellation of propagation terms. However, remember that the way we have chosen our reference medium (in this example) means that group velocity corrections appear in the second RHS term as part of  $\tilde{\alpha}_c^D$ . Choosing a frame velocity equal to the group velocity here will leave a residual reference-like term, but this can then cancel with part of the group-velocity-like contribution from the dispersion term, which in most cases would give the best cancellation – just as in the usual  $E$  field approaches. If we wanted, we could easily rearrange eqn.(4.5.7) to incorporate such a cancellation, and then solve the equation appropriately.



## 4. The dispersive correction

The next step is solving for the linear dispersion  $\tilde{\alpha}_c^D = \tilde{\epsilon}_c^D / \tilde{\alpha}_r$  in the frequency domain. Fortunately, this part of the equation is (also) trivial to solve exactly in the frequency domain, and results in the following exponential solution –

$$G_{xD}^\pm(z + \delta z) = G_x^\pm(z) \times \exp \left[ \mp i k G_x^\pm \cdot \delta z \mp \frac{i \omega \tilde{\epsilon}_c^D(\omega)}{2} \sqrt{\frac{\mu_0}{\epsilon_0}} [G_x^+(z) + G_x^-(z)] \cdot \delta z \right]. \quad (4.5.8)$$

Although both reference and dispersion steps can be solved exactly using exponentials, there is an important difference. The reference evolution of  $G^+$  depends only on  $G^+$ , whereas the dispersion evolution depends on the sum  $G^+ + G^-$ , since the dispersion acts on the electric field. In the forward-only approximation, of course,  $G^- = 0$  and the two steps can be trivially combined – as is automagically done in most approaches solving for the propagation of optical pulses.

## 5. The nonlinear correction

The final (third) step is to transform into the time domain and solve for the  $n$ -th order nonlinear effects. Since our reference  $\alpha_r, \beta_r$  are constants,  $\tilde{\alpha}_c^{NL} = \tilde{\epsilon}_c^{NL} / \tilde{\alpha}_r$  is trivial to calculate and will be just  $\tilde{\alpha}_c^{NL} = \chi^{(n)} E^{n-1} / \tilde{\alpha}_r$ , so (for a simple Euler method integration)

$$G_x^\pm(z + \delta z) = G_{xD}^\pm(z + \delta z) \pm \frac{\chi^{(n)}}{2} \sqrt{\frac{\mu_0}{\epsilon_0}} \frac{d}{dt} [G_{xD}^+(z + \delta z) + G_{xD}^-(z + \delta z)]^n \delta z \quad (4.5.9)$$

For a narrow-band field, the time derivative would be dominated by (proportional to) its centre frequency; and indeed in most envelope theories we only see a factor of (carrier frequency)  $\omega_0$  in the analogous expression. For a wider-band field, there would be corrections, as found for the GFEA approach [7].

In an envelope theory, the time derivative would be dominated by the carrier frequency  $\omega_0$ , hence we would only see a factor of  $\omega_0$  rather than the differential here. A few-cycle envelope theory would give us corrections to the  $\omega_0$  prefactor. If you wanted to, the derivative used in this nonlinear step could be calculated in the spectral domain – you'd have to FT  $G$  into the time domain, raise it to the  $n$ -th power, and FT back...

## 6. Initial conditions: matching a pulse to the medium

Here I describe how to make the “best matched” initial conditions for  $G^\pm$  describing a pulse propagating *only* in the “+” direction. Note that it may be helpful to read section IX if you are confused by the use of terms like “pseudo-reflection” or by the meaning of the co-propagating  $G^-$  component.

The procedure is:

- (1) Pick a suitable electric field  $E(\omega)$
- (2) Find out the full dispersive properties of the medium, as described by  $\epsilon_i$  and  $\mu_i$ . You can even put the nonlinear or any other properties into  $\epsilon_i$  and  $\mu_i$  as well – but only if you will still know how get a solution for steps (3) and (5) below with those added complications.
- (3) Calculate the  $H(\omega)$  corresponding to  $E(\omega)$  for a “+” propagating pulse:

$$0 = \sqrt{\epsilon_i(\omega)} E(\omega) + \sqrt{\mu_i(\omega)} H(\omega) \quad (4.5.10)$$

$$\Rightarrow H(\omega) = \sqrt{\frac{\epsilon_i(\omega)}{\mu_i(\omega)}} E(\omega). \quad (4.5.11)$$

- (4) Choose our reference medium parameters  $\epsilon_r(\omega)$  and  $\mu_r(\omega)$ .



(5) Calculate our initial  $G^\pm$  for the chosen reference medium, given our initial  $E(\omega)$  and  $H(\omega)$  fields.

$$G^\pm = \sqrt{\epsilon_r(\omega)}E(\omega) \pm \sqrt{\mu_r(\omega)}H(\omega) \quad (4.5.12)$$

$$= \sqrt{\epsilon_r(\omega)}E(\omega) \pm \sqrt{\mu_r(\omega)}\sqrt{\frac{\epsilon_i(\omega)}{\mu_i(\omega)}}E(\omega) \quad (4.5.13)$$

$$= \left[ \sqrt{\epsilon_r(\omega)} \pm \sqrt{\mu_r(\omega)}\sqrt{\frac{\epsilon_i(\omega)}{\mu_i(\omega)}} \right] E(\omega). \quad (4.5.14)$$

These  $G^\pm$  initial conditions are the best guess we can make for a “+” propagating pulse in our medium.

Note that in step (3) above, I have (in effect) used the definition of  $G^\pm$  with a “reference” equal to the full dispersive properties of the medium, and achieved a “+” propagating pulse by setting  $G^-$  to zero. In the case where we intend to use these full dispersive properties as or simulation reference, of course step (5) becomes trivial.

Because our reference  $\epsilon_r, \mu_r$  will not (in general) include all the dispersion  $\epsilon_i(\omega), \mu_i(\omega)$ , our initial conditions will contain both a  $G^+$  and a  $G^-$  part. This initial  $G^-$  part will co-propagate along with the  $G^+$ , the combination being a (hopefully) good match to the medium.

If our “full” dispersion  $\epsilon_i(\omega), \mu_i(\omega)$  are not a perfect match to the simulated medium, then there will be an initial pseudo-reflection. This pseudo-reflection will consist of a reverse propagating  $G^-$ , whose size will be equivalent to that of the reflection from an interface between the dispersion used to construct the initial conditions and the simulated medium (with  $\epsilon_r + \epsilon_c, \mu_r + \mu_c$ ). Mostly this kind of pseudo-reflection can be eliminated with appropriate choice of  $\epsilon_i(\omega), \mu_i(\omega)$ ; although nonlinear corrections will usually be impossible to treat, and so will still cause pseudo-reflections – albeit hopefully very small ones.

For example, if we construct our initial conditions using a vacuum  $\epsilon_i(\omega) = \epsilon_0, \mu_i(\omega) = \mu_0$ , but simulate a medium with some refractive index  $n$ , this will be equivalent to simulating a  $\epsilon_i, \mu_i \rightarrow \epsilon_r, \mu_r$  interface. Such an interface will cause a (pseudo) reflection, which will consist of a reverse propagating  $G^-$ , whose size will be equivalent to that of the reflection from a vacuum/ $n$  interface. A mirror image of the pseudo-reflection will also appear, and co-propagate with the  $G^+$  pulse – to ensure that the transmitted  $E$  field has reduced appropriately for such an interface.

Note that the “forward (co) propagating”  $G^-$  component still has a Poynting vector directed *backwards*.

## 7. Comparing alternate references

Here I do the simplest possible comparison between two different implementations of the same physical system; and see that (within the limits set by the approximations) that they are identical. In the uncoupled stationary frame ( $G^+$  only) case, the wave equation with only a reference  $\alpha'_0$  term is

$$\partial_z G_x^\pm = \mp i\omega \alpha'_0 \beta_r G_x^+ \quad (4.5.15)$$

$$= \mp i\omega [\alpha_r^2 + \alpha_r \alpha_L]^{1/2} \beta_r G_x^+ \quad (4.5.16)$$

$$= \mp i\omega \alpha_r \beta_r [1 + \alpha_L/\alpha_r]^{1/2} G_x^+ \quad (4.5.17)$$

$$\approx \mp i\omega \alpha_r \beta_r \left[ 1 + \frac{1}{2} \alpha_L/\alpha_r \right] G_x^+ \quad (4.5.18)$$

$$= \mp i\omega \left[ \alpha_r \beta_r + \frac{1}{2} \alpha_L \beta_r \right] G_x^+ \quad (4.5.19)$$

$$= \mp i\omega \alpha_r \beta_r G_x^+ \mp \frac{i\omega \alpha_L \beta_r}{2} G_x^+ \quad (4.5.20)$$

This is identical to the equivalent wave equation with both a reference and correction term, with the two sets of  $\alpha$  parameters linked by the appropriate relation  $\epsilon = \alpha'^2_0 = \alpha_r^2 + \alpha_r \alpha_L$ . The uncoupled approximation (no  $G^-$  i.e.  $|G^-| \ll |G^+|$ ) and small correction ( $1 \gg \alpha_L/\alpha_r$ ) are true in the same regime.



## V. FIRST ORDER EVOLUTION EQUATIONS (PRIMARY $\vec{H}$ )

If we were more interested in  $\vec{H}$  than  $\vec{E}$ , we might instead define the field variables as in III D, using eqn (3.4.1, 3.4.4, 3.4.5). So

$$\vec{G}'^\pm(\omega) = \beta_r(\omega) \vec{H}(\omega) \pm \vec{u} \times \alpha_r(\omega) \vec{E}(\omega), \quad (5.0.1)$$

$$\text{so } \vec{u} \times \alpha_r \vec{E} = \frac{1}{2} [\vec{G}'^+ - \vec{G}'^-] \quad (5.0.2)$$

$$\text{and } \beta_r \vec{H} = \frac{1}{2} [\vec{G}'^+ + \vec{G}'^-]. \quad (5.0.3)$$

### A. Vector stationary frame (Authoritative)

Here I avoid calculations using the  $x, y$ , and  $z$  components of the field, and retain a fully vector description. Using the definitions in subsection II, I can proceed with the calculation –

$$\begin{aligned} \nabla \times \vec{H}(t) &= +\partial_t \epsilon * \vec{E}(t) + \vec{J}, \\ \nabla \times \vec{E}(t) &= -\partial_t \mu * \vec{H}(t) \end{aligned} \quad (5.1.1)$$

into temporal-frequency space —→

$$\nabla \times \vec{H}(\omega) = -i\omega \alpha(\omega)^2 * \vec{E}(\omega) + \vec{J}(\omega),$$

$$\nabla \times \vec{E}(\omega) = +i\omega \beta(\omega)^2 * \vec{H}(\omega) \quad (5.1.2)$$

swap transverse components —→

$$\nabla \times \vec{H} = -i\omega \alpha^2 * \vec{E} + \vec{J},$$

$$\vec{u} \times (\nabla \times \vec{E}) = +i\omega \beta^2 * (\vec{u} \times \vec{H}) \quad (5.1.3)$$

$$\text{premultiply} \longrightarrow \nabla \times \beta_r \vec{H} = -i\omega \beta_r \alpha^2 * \vec{E} + \beta_r \vec{J},$$

$$\vec{u} \times (\nabla \times \alpha_r \vec{E}) = +i\omega \alpha_r \beta^2 * (\vec{u} \times \vec{H}) \quad (5.1.4)$$

$$\text{sum-and-difference} \longrightarrow \nabla \times \beta_r \vec{H} \pm \vec{u} \times (\nabla \times \alpha_r \vec{E}) = -i\omega \beta_r \alpha^2 * \vec{E} \pm i\omega \alpha_r \beta^2 * (\vec{u} \times \vec{H}) + (5.34 \vec{J})$$

The vector  $\vec{G}'^\pm$  fields (defined in eqn (3.4.1)) are

$$\vec{G}'^\pm = \beta_r \vec{H} \pm \vec{u} \times \alpha_r \vec{E} \quad (5.1.6)$$

This means I need to convert both the second term on the LHS of the sum-and-difference equation above, as well as the RHS. It is most important for the LHS to be simple, because this will define the type of propagation specified by the RHS. To convert the LHS, we need two vector identities, as used in subsection XII to get eqn.(12.0.10). So using

$$\vec{u} \times (\nabla \times \vec{H}) - \nabla (\vec{u} \cdot \vec{H}) = \nabla \times (\vec{u} \times \vec{H}) \quad (5.1.7)$$

If I retain the  $\vec{u} \cdot \alpha_r \vec{D}$  term, which will be zero for strictly transverse  $D$  fields, then –

$$\nabla \times \beta_r \vec{H} \pm \vec{u} \times (\nabla \times \alpha_r \vec{E}) = -i\omega \beta_r \alpha^2 * \vec{E} \pm i\omega \alpha_r \beta^2 * (\vec{u} \times \vec{H}) + \beta_r \vec{J}, \quad (5.1.8)$$

$$\nabla \times \beta_r \vec{H} \pm \nabla \times (\vec{u} \times \alpha_r \vec{E}) \pm \nabla (\vec{u} \cdot \alpha_r \vec{E}) = -i\omega \beta_r \alpha^2 * \vec{E} \pm i\omega \alpha_r \beta^2 * (\vec{u} \times \vec{H}) + \beta_r \vec{J} \quad (5.1.9)$$

$$\nabla \times [\beta_r \vec{H} \pm (\vec{u} \times \alpha_r \vec{E})] = -i\omega \beta_r \alpha^2 * \vec{E} \pm i\omega \alpha_r \beta^2 * (\vec{u} \times \vec{H}) \mp \nabla (\vec{u} \cdot \alpha_r \vec{E}) + \beta_r \vec{J} \quad (5.1.10)$$

$$\nabla \times \vec{G}'^\pm = -i\omega \{\beta_r \alpha^2 * \vec{E} \mp \alpha_r \beta^2 * (\vec{u} \times \vec{H})\} \mp \nabla (\vec{u} \cdot \alpha_r \vec{E}) + \beta_r \vec{J} \quad (5.1.11)$$

Now note that eqn.(12.0.12) means that

$$\vec{u} \times [\vec{u} \times \vec{H}] = [\vec{u} \cdot \vec{H}] \vec{u} - \vec{H} \quad (5.1.12)$$

$$\text{and so } \vec{u} \times \vec{G}'^\pm = \vec{u} \times \beta_r \vec{H} \pm \vec{u} \times [\vec{u} \times \alpha_r \vec{E}] \quad (5.1.13)$$

$$= \vec{u} \times \beta_r \vec{H} \mp \alpha_r \vec{E} \pm [\vec{u} \cdot \alpha_r \vec{E}] \vec{u} \quad (5.1.14)$$



So,

$$\nabla \times \vec{G}'^\pm = -\omega \left\{ \beta_r \alpha^2 * [\vec{u} \cdot \vec{E}] \vec{u} - \beta_r \alpha^2 * (\vec{u} \times [\vec{u} \times \vec{E}]) \mp \alpha_r \beta^2 * [\vec{u} \times \vec{H}] \right\} \mp \nabla (\vec{u} \cdot \alpha_r \vec{E}) + \beta_r \vec{J}.15)$$

$$= +\omega \left\{ \beta_r \alpha^2 * \vec{u} \times [\vec{u} \times \vec{E}] \pm \alpha_r \beta^2 * [\vec{u} \times \vec{H}] \right\} - \omega \beta_r \alpha^2 * [\vec{u} \cdot \vec{E}] \vec{u} \mp \nabla (\vec{u} \cdot \alpha_r \vec{E}) + \beta_r \vec{J}.16)$$

I now separate the interaction parts (depending on  $\alpha_c$ ,  $\beta_c$ ) from the reference parts (depending on  $\alpha_r$ ,  $\beta_r$ ), and then substitute (as far as possible) expressions containing  $G'^\pm$  rather than  $E$  or  $H$ , by referring to eqns (3.4.4) and (3.4.5). Hence

$$\begin{aligned} \nabla \times \vec{G}'^\pm &= +\omega \alpha_r \beta_r \left\{ \vec{u} \times [\vec{u} \times \alpha_r \vec{E}] \pm [\vec{u} \times \beta_r \vec{H}] \right\} - \omega \alpha_r \beta_r [\vec{u} \cdot \alpha_r \vec{E}] \vec{u} \mp \nabla (\vec{u} \cdot \alpha_r \vec{E}) \\ &\quad + \omega \beta_r \alpha_c * (\vec{u} \times [\vec{u} \times \alpha_r \vec{E}]) \pm \omega \alpha_r \beta_c * [\vec{u} \times \beta_r \vec{H}] - \omega \beta_r \alpha_c * [\vec{u} \cdot \alpha_r \vec{E}] \vec{u} + \beta_r \vec{J} \end{aligned} \quad (5.1.17)$$

$$\begin{aligned} &= \pm \omega \alpha_r \beta_r \vec{u} \times \vec{G}'^\pm - \omega \alpha_r \beta_r \vec{u} G'^\circ \mp \nabla G'^\circ \\ &\quad + \frac{\omega \beta_r}{2} \alpha_c * (\vec{u} \times [\vec{G}'^+ + \vec{G}'^-]) \pm \frac{\omega \alpha_r}{2} \beta_c * (\vec{u} \times [\vec{G}'^+ - \vec{G}'^-]) - \omega \beta_r \alpha_c * (\vec{u} G'^\circ) + \beta_r \vec{J}.18) \end{aligned}$$

Here the transverse (TEF) and longitudinal parts decouple, since  $\vec{G}'^\pm$  is guaranteed electrically transverse, and  $\vec{u} G'^\circ$  is guaranteed electrically longitudinal. In any case, the two decoupled equations are –

$$\nabla \times \vec{G}'^\pm = \pm \omega \alpha_r \beta_r \vec{u} \times \vec{G}'^\pm + \frac{\omega \beta_r}{2} \alpha_c * (\vec{u} \times [\vec{G}'^+ + \vec{G}'^-]) \pm \frac{\omega \alpha_r}{2} \beta_c * (\vec{u} \times [\vec{G}'^+ - \vec{G}'^-]) + \beta_r \vec{J} \quad (5.1.19)$$

$$\pm \nabla G'^\circ = -\omega \alpha_r \beta_r \vec{u} G'^\circ - \omega \alpha_r \beta_c * (\vec{u} G'^\circ). \quad (5.1.20)$$

We see from the following VA 1 that these two equations are sufficient to describe the fields as long as there is no longitudinal  $\vec{H}$  component (i.e.  $\vec{G}'^\pm$  is transverse); (also with reference to the divergence calculation in III D 1).

### 1. Longitudinal Equation

Taking the  $\vec{u} \times$  of the  $\nabla \times \vec{E}$  equation removed a part of the field dynamics lying in the direction of the propagation vector  $\vec{u}$ . To rectify this omission, we take the dot product with  $\vec{u}$ . Using the standard vector identity eqn.(12.0.13):

$$\nabla \cdot (\vec{A} \times \vec{B}) = \vec{A} \cdot (\nabla \times \vec{B}) - \vec{B} \cdot (\nabla \times \vec{A}), \quad (5.1.21)$$

and using  $\nabla \times \vec{u} = 0$ ; we get

$$\vec{u} \cdot (\nabla \times \vec{E}) = -\omega \beta^2 * (\vec{u} \cdot \vec{H}) \quad (5.1.22)$$

$$\text{vector identity: } \nabla \cdot (\vec{u} \times \vec{E}) = -\omega \beta^2 * (\vec{u} \cdot \vec{H}) \quad (5.1.23)$$

$$\text{premultiply: } \beta_r \nabla \cdot (\vec{u} \times \alpha_r \vec{E}) = -\omega \alpha_r \beta^2 * (\vec{u} \cdot \beta_r \vec{H}) \quad (5.1.24)$$

$$\alpha_r \nabla \cdot (\vec{G}'^+ - \vec{G}'^-) = -\omega \alpha_r \beta^2 * (\vec{u} \cdot [\vec{G}'^+ + \vec{G}'^-]) \quad (5.1.25)$$

This is the same as the divergence-difference equation seen in III D 1; where we see that both parts of the LHS are zero if  $\vec{u} \cdot (\vec{G}'^+ + \vec{G}'^-) = 0$ .

### 2. Simple Cases

In the non-magnetic ( $\beta_c = 0$ ) case,

$$\nabla \times \vec{G}'^\pm = \pm \omega \alpha_r \beta_r \vec{u} \times \vec{G}'^\pm + \frac{\omega \beta_r}{2} \alpha_c * (\vec{u} \times [\vec{G}'^+ + \vec{G}'^-]) + \beta_r \vec{J} \quad (5.1.26)$$

$$\pm \nabla G'^\circ = -\omega \alpha_r \beta_r \vec{u} G'^\circ. \quad (5.1.27)$$



In the time domain, eqn.(5.1.26) is

$$\nabla \times \vec{G}'^\pm = \mp \partial_t \left[ (\alpha_r * \beta_r) * \left( \vec{u} \times \vec{G}'^\pm \right) \right] - \partial_t \left[ \left( \frac{\beta_r}{2} \right) * \alpha_c * \left( \vec{u} \times [\vec{G}^+ + \vec{G}^-] \right) \right] \pm \vec{u} \times \beta_r * \vec{J}. \quad (5.1.28)$$

### Plane Polarized & Magnetic ( $G_x^\pm \sim E_x \pm H_y$ ):

The  $\hat{y}$  components of the LHS and RHS contain the  $G_x$  values – as a result of the curl and cross-products respectively. Compare the previous stand-alone calculation in eqn.(4.1.8) to

$$\partial_z G'_x \pm = \mp i\omega \alpha_r \beta_r G'_x \pm \mp \frac{i\omega \beta_r}{2} \alpha_c * [G'_x^+ + G'_x^-] - \frac{i\omega \alpha_r}{2} \beta_c * [G'_x^+ - G'_x^-] \quad (5.1.29)$$

### Vector Moving Frame, Transverse, Magnetic:

It is worth transforming to the moving frame at this end point, rather than at the beginning. Since eqn.(3.8.5) tells us that  $\nabla \vec{Q} = \nabla' \vec{Q} - \alpha_r \beta_r \vec{u} \times \partial_t' \vec{Q}$ , with  $\xi = \alpha_r \beta_r / \alpha_r \beta_r$ , we get

$$\begin{aligned} \nabla' \times \vec{G}'^\pm + i\omega \alpha_r \beta_r \vec{u} \times \vec{G}'^\pm &= \pm i\omega \alpha_r \beta_r \vec{u} \times \vec{G}^\pm + \frac{i\omega \beta_r}{2} \alpha_c * \left( \vec{u} \times [\vec{G}'^+ + \vec{G}'^-] \right) \mp \frac{i\omega \alpha_r}{2} \beta_c * \left( \vec{u} \times [\vec{G}'^+ - \vec{G}'^-] \right) \\ \nabla' \times \vec{G}^\pm &= \pm i\omega \alpha_r \beta_r (1 \mp \xi) \vec{u} \times \vec{G}'^\pm + \frac{i\omega \beta_r}{2} \alpha_c * \left( \vec{u} \times [\vec{G}'^+ + \vec{G}'^-] \right) \mp \frac{i\omega \alpha_r}{2} \beta_c * \left( \vec{u} \times [\vec{G}'^+ - \vec{G}'^-] \right) \end{aligned} \quad (5.1.30)$$

### Plane Polarized Moving Frame, Magnetic:

This is just a special case of the above vector moving frame equation –

$$\partial_{z'} G'_x \pm = \pm i\omega \alpha_r \beta_r (1 \mp \xi) G_x^\pm + \frac{i\omega \beta_r}{2} \alpha_c * [G_x^+ + G_x^-] \mp \frac{i\omega \alpha_r}{2} \beta_c * [G_x^+ - G_x^-] \quad (5.1.32)$$



## VI. FIRST ORDER EVOLUTION EQUATIONS (PRIMARY $\vec{D}$ )

Follows the derivation in the previous section, but uses the “Primary  $\vec{D}$ ” (or  $\vec{F}^\pm$ ) form, as defined in III C, with eqn. (3.5.1, 3.5.3, 3.5.5). So (dropping the tildes out of laziness)

$$\vec{F}^\pm(\omega) = \alpha_r^{-1}(\omega) \vec{D}(\omega) \pm \vec{u} \times \beta_r^{-1}(\omega) \vec{B}(\omega), \quad (6.0.1)$$

$$\text{so} \quad \alpha_r^{-1} \vec{D} = \frac{1}{2} [\vec{F}^+ + \vec{F}^-] \quad (6.0.2)$$

$$\text{and} \quad \vec{u} \times \beta_r^{-1} \vec{B} = \frac{1}{2} [\vec{F}^+ - \vec{F}^-]. \quad (6.0.3)$$

We would do this so that the displacement field (and magnetic fields) could be easily reconstructed from  $\vec{F}^\pm$ ; and, in analogy to the THF and TEF mentioned above, we would use a transverse magnetic induction  $B$  approximation (TBF). This this form differs from the primary  $E$  form that defines  $G^\pm$  in that it contains  $\alpha_r^{-1}D = \alpha_r^{-1}\alpha^2E$  not  $\alpha_rE$ ; i.e. the electric-like contribution differs only by  $\epsilon/\epsilon_r = \alpha^2/\alpha_r^2$ . A similar comparison holds for the magnetic components.

### A. Vector stationary frame (Authoritative)

Here I avoid calculations using the  $x, y$ , and  $z$  components of the field, and retain a fully vector description. Using the definitions in subsection II, I can proceed with the calculation –

$$\begin{aligned} \nabla \times \vec{H}(t) &= +\partial_t \vec{D}(t) + \vec{J}, \\ \nabla \times \vec{E}(t) &= -\partial_t \vec{B}(t) \end{aligned} \quad (6.1.1)$$

$$\begin{aligned} \text{into temporal-frequency space} \longrightarrow \quad \nabla \times \vec{H}(\omega) &= -i\omega \vec{D}(\omega) + \vec{J}(\omega), \\ \nabla \times \vec{E}(\omega) &= +i\omega \vec{B}(\omega) \end{aligned} \quad (6.1.2)$$

$$\begin{aligned} \text{swap transverse components} \longrightarrow \quad \vec{u} \times (\nabla \times \vec{H}) &= -i\omega \vec{u} \times \vec{D} + \vec{u} \times \vec{J}, \\ \nabla \times \vec{E} &= +i\omega \vec{B} \end{aligned} \quad (6.1.3)$$

$$\begin{aligned} \text{premultiply} \longrightarrow \quad \vec{u} \times (\nabla \times \alpha_r \beta^2 * \vec{H}) &= -i\omega \alpha_r \beta^2 * (\vec{u} \times \vec{D}) + \vec{u} \times \alpha_r \beta^2 * \vec{J}, \\ \nabla \times \beta_r \alpha^2 * \vec{E} &= +i\omega \beta_r \alpha^2 * \vec{B} \end{aligned} \quad (6.1.4)$$

$$\begin{aligned} \text{simplify} \longrightarrow \quad \vec{u} \times (\nabla \times \alpha_r \vec{B}) &= -i\omega \alpha_r \beta^2 * (\vec{u} \times \vec{D}) + \vec{u} \times \alpha_r \beta^2 * \vec{J}, \\ \nabla \times \beta_r \vec{D} &= +i\omega \beta_r \alpha^2 * \vec{B} \end{aligned} \quad (6.1.5)$$

$$\begin{aligned} \text{sum-and-difference} \longrightarrow \quad \nabla \times \beta_r \vec{D} \pm \vec{u} \times (\nabla \times \alpha_r \vec{B}) &= +i\omega \beta_r \alpha^2 * \vec{B} \mp i\omega \alpha_r \beta^2 * (\vec{u} \times \vec{D}) \\ &\pm \vec{u} \times \alpha_r \beta^2 * \vec{J}, \end{aligned} \quad (6.1.6)$$

The vector  $\vec{F}^\pm$  fields (defined in eqn (3.5.1)) are

$$\vec{F}^\pm = \alpha_r^{-1} \vec{D} \pm \vec{u} \times \beta_r^{-1} \vec{B} \quad (6.1.7)$$

This means I need to convert both the second term on the LHS of the sum-and-difference equation above, as well as the RHS. It is most important for the LHS to be simple, because this will define the type of propagation specified by the RHS. To convert the LHS, we need two vector identities, as used in subsection XII to get eqn.(12.0.10). So using

$$\vec{u} \times (\nabla \times \vec{H}) - \nabla (\vec{u} \cdot \vec{H}) = \nabla \times (\vec{u} \times \vec{H}) \quad (6.1.8)$$



If I retain the  $\vec{u} \times \beta_r \vec{H}$  term in the above, which will be zero for strictly transverse fields, then –

$$\nabla \times \beta_r \vec{D} \pm \vec{u} \times (\nabla \times \alpha_r \vec{B}) = +\omega \beta_r \alpha^2 * \vec{B} \mp \omega \alpha_r \beta^2 * (\vec{u} \times \vec{D}) \pm \vec{u} \times \alpha_r \beta^2 * \vec{J} \quad (6.1.9)$$

$$\nabla \times \beta_r \vec{D} \pm \nabla \times (\vec{u} \times \alpha_r \vec{B}) \pm \nabla (\vec{u} \cdot \alpha_r \vec{B}) = +\omega \beta_r \alpha^2 * \vec{B} \mp \omega \alpha_r \beta^2 * (\vec{u} \times \vec{D}) \pm \vec{u} \times \alpha_r \beta^2 * \vec{J} \quad (6.1.10)$$

$$\begin{aligned} \nabla \times [\beta_r \vec{D} \pm (\vec{u} \times \alpha_r \vec{B})] &= +\omega \beta_r \alpha^2 * \vec{B} \mp \omega \alpha_r \beta^2 * (\vec{u} \times \vec{D}) \mp \nabla (\vec{u} \cdot \alpha_r \vec{B}) \\ &\quad \pm \vec{u} \times \alpha_r \beta^2 * \vec{J} \end{aligned} \quad (6.1.11)$$

$$\nabla \times \vec{F}^\pm = \omega \{ \beta_r \alpha^2 * \vec{B} \mp \alpha_r \beta^2 * (\vec{u} \times \vec{D}) \} \mp \nabla (\vec{u} \cdot \alpha_r \vec{B}) \pm \vec{u} \times \alpha_r \beta^2 * \vec{J} \quad (6.1.12)$$

Now note that eqn.(12.0.12) means that

$$\vec{u} \times [\vec{u} \times \vec{H}] = [\vec{u} \cdot \vec{H}] \vec{u} - \vec{H} \quad (6.1.13)$$

$$\text{and so } (\alpha_r \beta_r) \vec{u} \times \vec{F}^\pm = \vec{u} \times \beta_r \vec{D} \pm \vec{u} \times [\vec{u} \times \alpha_r \vec{B}] \quad (6.1.14)$$

$$= \vec{u} \times \beta_r \vec{D} \mp \alpha_r \vec{B} \pm [\vec{u} \cdot \alpha_r \vec{B}] \vec{u} \quad (6.1.15)$$

So,

$$\begin{aligned} (\alpha_r \beta_r) \nabla \times \vec{F}^\pm &= \omega \{ \beta_r \alpha^2 * [\vec{u} \cdot \vec{B}] \vec{u} - \beta_r \alpha^2 * \vec{u} \times [\vec{u} \times \vec{B}] \mp \alpha_r \beta^2 * (\vec{u} \times \vec{D}) \} \mp \nabla (\vec{u} \cdot \alpha_r \vec{B}) \\ &\quad \pm \vec{u} \times \alpha_r \beta^2 * \vec{J} \end{aligned} \quad (6.1.16)$$

$$\begin{aligned} &= \mp \omega \{ \alpha_r \beta^2 \vec{u} \times \vec{D} \pm \beta_r \alpha^2 \vec{u} \times [\vec{u} \times \vec{B}] \} + \omega \beta_r \alpha^2 [\vec{u} \cdot \vec{B}] \vec{u} \mp \nabla (\vec{u} \cdot \alpha_r \vec{B}) \\ &\quad \pm \vec{u} \times \alpha_r \beta^2 * \vec{J} \end{aligned} \quad (6.1.17)$$

I now separate the interaction parts (depending on  $\alpha_c, \beta_c$ ) from the reference parts (depending on  $\alpha_r, \beta_r$ ), and then substitute (as far as possible) expressions containing  $F$  rather than  $D$  or  $B$ , by referring to eqns (3.5.3) and (3.5.5). Hence

$$\begin{aligned} \nabla \times \vec{F}^\pm &= \mp \omega \alpha_r \beta_r \{ \vec{u} \times \alpha_r^{-1} \vec{D} \pm \vec{u} \times [\vec{u} \times \beta_r^{-1} \vec{B}] \} + \omega \alpha_r \beta_r [\vec{u} \cdot \beta_r^{-1} \vec{B}] \vec{u} \mp \nabla (\vec{u} \cdot \beta_r^{-1} \vec{B}) \\ &\quad \mp \omega \alpha_r \beta_c * (\vec{u} \times \alpha_r^{-1} \vec{D}) - \omega \beta_r \alpha_c * (\vec{u} \times [\vec{u} \times \beta_r^{-1} \vec{B}]) + \omega \beta_r \alpha_c * [\vec{u} \cdot \beta_r^{-1} \vec{B}] \vec{u} \\ &\quad \pm \vec{u} \times \beta_r^{-1} (\beta_r^2 + \beta_r \beta_c) * \vec{J} \end{aligned} \quad (6.1.18)$$

$$\begin{aligned} &= \mp \omega \alpha_r \beta_r \vec{u} \times \vec{F}^\pm + \omega \alpha_r \beta_r \vec{u} F^\circ \mp \nabla F^\circ \\ &\quad \mp \frac{\omega \alpha_r}{2} \beta_c * (\vec{u} \times [\vec{F}^+ + \vec{F}^-]) - \frac{\omega \beta_r}{2} \alpha_c * (\vec{u} \times [\vec{F}^+ - \vec{F}^-]) + \omega \beta_r \alpha_c * (\vec{u} F^\circ) \\ &\quad \pm \vec{u} \times (\beta_r + \beta_c) * \vec{J} \end{aligned} \quad (6.1.19)$$

Here the transverse (TDF) and longitudinal parts decouple, since  $\vec{F}^\pm$  is guaranteed displacement transverse, and  $\vec{u} F^\circ$  is displacement longitudinal. Note that in this *transverse/ longitudinal split*, the role of the current  $\vec{J}$  has not been checked. In any case, the two decoupled equations are –

$$\nabla \times \vec{F}^\pm = \mp \omega \alpha_r \beta_r \vec{u} \times \vec{F}^\pm \mp \frac{\omega \alpha_r}{2} \beta_c * (\vec{u} \times [\vec{F}^+ + \vec{F}^-]) - \frac{\omega \beta_r}{2} \alpha_c * (\vec{u} \times [\vec{F}^+ - \vec{F}^-]) \pm \vec{u} \times (\beta_r + \beta_c) * \vec{J} \quad (6.1.20)$$

$$\pm \nabla F^\circ = +\omega \alpha_r \beta_r \vec{u} F^\circ + \omega \beta_r \alpha_c * (\vec{u} F^\circ). \quad (6.1.21)$$

We see from the following VI A 1 that these two equations are sufficient to describe the fields as long as there is no longitudinal  $\vec{D}$  component (i.e.  $\vec{F}^\pm$  is transverse) or longitudinal  $\vec{J}$  (also with reference to the divergence calculation in III E 1).

### 1. Longitudinal Equation

Taking the  $\vec{u} \times$  of the  $\nabla \times \vec{H}$  equation removed a part of the field dynamics lying in the direction of the propagation vector  $\vec{u}$ . This part also include the response to longitudinal currents. To rectify this omission, we take the dot



product with  $\vec{u}$ . Using the standard vector identity eqn.(12.0.13):

$$\nabla \cdot (\vec{A} \times \vec{B}) = \vec{A} \cdot (\nabla \times \vec{B}) - \vec{B} \cdot (\nabla \times \vec{A}), \quad (6.1.22)$$

and using  $\nabla \times \vec{u} = 0$ ; we get

$$\vec{u} \cdot (\nabla \times \vec{H}) = -\omega \vec{u} \cdot \vec{D} + \vec{u} \cdot \vec{J} \quad (6.1.23)$$

$$\text{vector identity: } \nabla \cdot (\vec{u} \times \vec{H}) = -\omega \vec{u} \cdot \vec{D} + \vec{u} \cdot \vec{J} \quad (6.1.24)$$

$$\text{premultiply: } \alpha_r^{-1} \nabla \cdot (\vec{u} \times \beta_r^{-1} \vec{B}) = -\omega \beta_r^{-1} \beta^2 * (\vec{u} \cdot \alpha_r^{-1} \vec{E}) + \alpha_r^{-1} \beta_r^{-1} \beta^2 * (\vec{u} \cdot \vec{J}) \quad (6.1.25)$$

$$\nabla \cdot (\vec{F}^+ - \vec{F}^-) = -\omega \alpha_r (\beta_r + \beta_c) * (\vec{u} \cdot [\vec{F}^+ - \vec{F}^-]) + (\beta_r + \beta_c) * (\vec{u} \cdot \vec{J}) \quad (6.1.26)$$

This is the same as the divergence-difference equation seen in III E 1; where we see that both parts of the LHS are zero if  $\vec{u} \cdot (\vec{F}^+ + \vec{F}^-) = 0$  and  $\vec{u} \cdot \vec{J} = 0$ .

## 2. Simple Cases

In the non-magnetic ( $\beta_c = 0$ ) case,

$$\nabla \times \vec{F}^\pm = \mp \omega \alpha_r \beta_r \vec{u} \times \vec{F}^\pm - \frac{\imath \omega \beta_r}{2} \alpha_c * (\vec{u} \times [\vec{F}^+ - \vec{F}^-]) \pm \vec{u} \times (\beta_r + \beta_c) * \vec{J} \quad (6.1.27)$$

$$\pm \nabla F^\circ = +\omega \alpha_r \beta_r \vec{u} \cdot F^\circ + \imath \omega \beta_r \alpha_c * (\vec{u} \cdot F^\circ). \quad (6.1.28)$$

In the time domain, eqn.(6.1.27) is

$$\nabla \times \vec{F}^\pm = \pm \partial_t [(\alpha_r * \beta_r) * (\vec{u} \times \vec{F}^\pm)] + \partial_t \left[ \left( \frac{\beta_r}{2} \right) * \alpha_c * (\vec{u} \times [\vec{F}^+ - \vec{F}^-]) \right] \pm \vec{u} \times (\beta_r + \beta_c) * \vec{J}. \quad (6.1.29)$$

### Plane Polarized & Magnetic ( $F_x^\pm \sim_x \pm B_y$ ):

The  $\vec{y}$  components of the LHS and RHS contain the  $F_x$  values – as a result of the curl and cross-products respectively. Compare the previous stand-alone calculation in eqn.(4.1.8) to

$$\partial_z F_x^\pm = \mp \imath \omega \alpha_r \beta_r F_x^\pm - \frac{\imath \omega \beta_r}{2} \alpha_c * [F_x^+ - F_x^-] \mp \frac{\imath \omega \alpha_r}{2} \beta_c * [F_x^+ + F_x^-] \quad (6.1.30)$$

### Vector Moving Frame, Transverse, Magnetic:

It is worth transforming to the moving frame at this end point, rather than at the beginning. Since eqn.(3.8.5) tells us that  $\nabla \vec{Q} = \nabla' \vec{Q} - \alpha_r \beta_r \vec{u} \times \partial_t' \vec{Q}$ , with  $\xi = \alpha_r \beta_r / \alpha_r \beta_r$ , we get

$$\nabla' \times \vec{F}^\pm + \imath \omega \alpha_r \beta_r \vec{u} \times \vec{F}^\pm = \mp \imath \omega \alpha_r \beta_r \vec{u} \times \vec{F}^\pm \mp \frac{\imath \omega \alpha_r}{2} \beta_c * (\vec{u} \times [\vec{F}^+ + \vec{F}^-]) - \frac{\imath \omega \beta_r}{2} \alpha_c * (\vec{u} \times [\vec{F}^+ - \vec{F}^-]) \quad (6.1.31)$$

$$\begin{aligned} \nabla' \times \vec{F}^\pm &= \mp \imath \omega \alpha_r \beta_r (1 \mp \xi) \vec{u} \times \vec{F}^\pm \\ &\mp \frac{\imath \omega \alpha_r}{2} \beta_c * (\vec{u} \times [\vec{F}^+ + \vec{F}^-]) - \frac{\imath \omega \beta_r}{2} \alpha_c * (\vec{u} \times [\vec{F}^+ - \vec{F}^-]) \end{aligned} \quad (6.1.32)$$

### Plane Polarized Moving Frame, Magnetic:

This is just a special case of the above vector moving frame equation –

$$\partial_z' F_x^\pm = \mp \imath \omega \alpha_r \beta_r (1 \mp \xi) F_x^\pm \mp \frac{\imath \omega \alpha_r}{2} \beta_c * ([F_x^+ + F_x^-]) - \frac{\imath \omega \beta_r}{2} \alpha_c * ([F_x^+ - F_x^-]) \quad (6.1.33)$$

### Beltrami-like form:

This might be written

$$\{(\nabla \times) \pm \imath \omega \alpha_r \beta_r (\vec{u} \times)\} \vec{F}^\pm = \vec{W}^\pm, \quad (6.1.34)$$

where  $\vec{W}^\pm$  is a “source” term, but note that actually it will be some complicated function of  $\vec{G}^\pm$ .



## VII. FIRST ORDER EVOLUTION EQUATIONS (PRIMARY $\vec{B}$ )

Follows the derivation in the previous sections, but uses the “Primary  $\vec{B}$ ” (or  $\vec{F}'^\pm$ ) form, as defined in III F, with eqns. (3.6.1, 3.6.4, 3.6.5).

**NOTE:** Not yet calculated, but it will have many similarities to the preceding three derivations in sections IV, V, VI.

## VIII. THE ENERGY DENSITY AND THE POYNTING VECTOR

Here I discuss the role and calculation of the energy densities and Poynting vector for the EM field described in terms of the Primary  $\vec{E}$  ( $\vec{G}^\pm$ ) fields. While it turns out that the THF Poynting vector is always a simple function of the  $\vec{G}^\pm$  fields, the energy density is more complicated and depends on the degree of mismatch between the reference parameters and the true material parameters. Because of this difficulty of reconstructing  $\vec{H}$  from  $\vec{G}^\pm$ , all these calculations currently resort to restricting themselves to the THF case only (i.e.  $G^o = 0$ , see III C).

In practise it might well be simpler to just reconstruct the  $\vec{E}$  and  $\vec{B}$  fields from known  $\vec{G}^\pm$ , and use the traditional forms for energy density and Poynting vector rather than evaluate the complicated expressions derived in this section. Nevertheless, these complicated expression may provide insight into the nature of the  $\vec{G}^\pm$  fields, even if they are not used for computation.

Here I will restrict myself to the non-dispersive reference case, where  $\epsilon, \mu$  are constants independent of frequency. Without this, getting  $\vec{E}(t)$  from  $\vec{G}^\pm(t)$  requires a deconvolution to get the frequency-dependance of  $\epsilon, \mu$  out of  $\vec{G}^\pm(t)$ . This restriction can be avoided entirely by taking the formulae below to hold in the frequency (rather than time) domain.

I define  $\odot$ , denoting a convolving dot product, and  $\otimes$ , denoting a convolving cross product, so

$$\vec{A} \odot \vec{B} = \int d\omega' \vec{A}(\omega') \cdot \vec{B}(\omega - \omega'); \quad (8.0.1)$$

$$\vec{A} \otimes \vec{B} = \int d\omega' \vec{A}(\omega') \times \vec{B}(\omega - \omega'). \quad (8.0.2)$$

I've used

$$[A(t) * \vec{B}(t)] \cdot \vec{C}(t) \leftrightarrow [A(\omega) \vec{B}(\omega)] \odot \vec{C}(\omega) \quad (8.0.3)$$

$$[A(t) * \vec{B}(t)] \times \vec{C}(t) \leftrightarrow [A(\omega) \vec{B}(\omega)] \otimes \vec{C}(\omega). \quad (8.0.4)$$

To avoid cluttered equations, I will also use this shorthand notation

$$\vec{G}_\epsilon^\pm = \vec{G}^\pm / \alpha_r^{1/2}, \quad \vec{G}_\mu^\pm = \vec{G}^\pm / \beta_r^{1/2}. \quad (8.0.5)$$

### A. Energy Density

The standard expression for bulk energy density of the EM field in terms of the electric and magnetic fields can be easily converted into one in terms of  $\vec{G}^\pm$  in the THF case. In the following, I explicitly allow for the possibility of mismatched reference and material parameters (i.e.  $\epsilon \neq \epsilon_r$  etc).

1. *Miscellaneous Identities*

Where, at the transverse approximation:

$$(\vec{u} \times \vec{A}) \cdot (\vec{u} \times \vec{B}) = \vec{A}' \cdot (\vec{u} \times \vec{B}) = -\vec{A}' \cdot (\vec{B} \times \vec{u}) = -\vec{B} \cdot (\vec{u} \times \vec{A}') \quad (8.1.1)$$

$$= -\vec{B} \cdot (\vec{u} \times \vec{u} \times \vec{A}) = -\vec{B} \cdot (\vec{u} \times [\vec{u} \times \vec{A}]) \quad (8.1.2)$$

$$= +\vec{B} \cdot ([\vec{u} \times \vec{A}] \times \vec{u}) = +\vec{B} \cdot ([\vec{u} \cdot \vec{u}] \vec{A} - [\vec{u} \cdot \vec{A}] \vec{u}) \quad (8.1.3)$$

$$= +\vec{B} \cdot \vec{A} - [\vec{B} \cdot \vec{u}] [\vec{u} \cdot \vec{A}] \quad (8.1.4)$$

$$\text{transverse.} \Rightarrow (\vec{u} \times \vec{A}) \cdot (\vec{u} \times \vec{B}) \approx +\vec{B} \cdot \vec{A} = \vec{A} \cdot \vec{B} \quad (8.1.5)$$

The following brief calculation illustrates the non-trivial nature of getting a simple expression in the frequency domain –

$$\mathcal{U}_a = \epsilon * E \cdot E \quad (8.1.6)$$

$$= \left[ \int \epsilon(t') E(t-t') dt' \right] \cdot E(t) \quad (8.1.7)$$

$$\tilde{\mathcal{U}}_a = \int e^{-i\omega t} \left[ \int \epsilon(t') E(t-t') dt' \right] \cdot E(t) dt \quad (8.1.8)$$

$$= \int \epsilon(t') \left[ \int e^{-i\omega t} E(t-t') \cdot E(t) dt \right] dt' \quad (8.1.9)$$

$$= \int \epsilon(t') \tilde{F}(\omega, t') dt' \quad (8.1.10)$$

$$\text{where } \tilde{F}(\omega, t') = \int e^{-i\omega t} E(t-t') \cdot E(t) dt \quad (8.1.11)$$



## 2. Transverse Case

The time domain energy density calculation is (making use of eqn.(8.0.5)),

$$\mathcal{U} = \frac{1}{2}\epsilon * \vec{E} \cdot \vec{E} + \frac{1}{2}\mu * \vec{H} \cdot \vec{H} \quad (8.1.12)$$

$$\text{freq. domain } \Rightarrow 2\mathcal{U}(\omega) = (\epsilon \vec{E}) \odot \vec{E} + (\mu \vec{H}) \odot \vec{H} \quad (8.1.13)$$

$$8\mathcal{U}(\omega) = (\epsilon [\vec{G}_\epsilon^+ + \vec{G}_\epsilon^-]) \odot [\vec{G}_\epsilon^+ + \vec{G}_\epsilon^-] + (\mu \vec{u} \times [\vec{G}_\mu^+ + \vec{G}_\mu^-]) \odot (\vec{u} \times [\vec{G}_\mu^+ + \vec{G}_\mu^-]) \quad (8.1.14)$$

$$\begin{aligned} 8\mathcal{U}(\omega) &= (\epsilon \vec{G}_\epsilon^+) \odot \vec{G}_\epsilon^+ + (\epsilon \vec{G}_\epsilon^+) \odot \vec{G}_\epsilon^- + (\epsilon \vec{G}_\epsilon^-) \odot \vec{G}_\epsilon^+ + (\epsilon \vec{G}_\epsilon^-) \odot \vec{G}_\epsilon^- \\ &\quad + (\vec{u} \times \mu \vec{G}_\mu^+) \odot (\vec{u} \times \vec{G}_\mu^+) - (\vec{u} \times \mu \vec{G}_\mu^+) \odot (\vec{u} \times \vec{G}_\mu^-) \\ &\quad - (\vec{u} \times \mu \vec{G}_\mu^-) \odot (\vec{u} \times \vec{G}_\mu^+) + (\vec{u} \times \mu \vec{G}_\mu^-) \odot (\vec{u} \times \vec{G}_\mu^-) \end{aligned} \quad (8.1.15)$$

$$\begin{aligned} \text{transverse approx eqn(8.1.5). } \Rightarrow &\approx (\epsilon \vec{G}_\epsilon^+) \odot \vec{G}_\epsilon^+ + (\epsilon \vec{G}_\epsilon^+) \odot \vec{G}_\epsilon^- + (\epsilon \vec{G}_\epsilon^-) \odot \vec{G}_\epsilon^+ + (\epsilon \vec{G}_\epsilon^-) \odot \vec{G}_\epsilon^- \\ &\quad + (\mu \vec{G}_\mu^+) \odot \vec{G}_\mu^+ - (\mu \vec{G}_\mu^+) \odot \vec{G}_\mu^- \\ &\quad - (\mu \vec{G}_\mu^-) \odot \vec{G}_\mu^+ + (\mu \vec{G}_\mu^-) \odot \vec{G}_\mu^- \end{aligned} \quad (8.1.16)$$

$$\begin{aligned} &= (\epsilon \vec{G}_\epsilon^+) \odot \vec{G}_\epsilon^+ + (\mu \vec{G}_\mu^+) \odot \vec{G}_\mu^+ + (\epsilon \vec{G}_\epsilon^-) \odot \vec{G}_\epsilon^- + (\mu \vec{G}_\mu^-) \odot \vec{G}_\mu^- \\ &\quad + (\epsilon \vec{G}_\epsilon^+) \odot \vec{G}_\epsilon^- - (\mu \vec{G}_\mu^+) \odot \vec{G}_\mu^- + (\epsilon \vec{G}_\epsilon^-) \odot \vec{G}_\epsilon^+ - (\mu \vec{G}_\mu^-) \odot \vec{G}_\mu^+ \end{aligned} \quad (8.1.17)$$

$$\begin{aligned} 8\mathcal{U}(\omega) &= \left( \left[ \frac{\epsilon}{\alpha_r} \vec{G}^+ \right] \odot \left[ \frac{1}{\alpha_r} \vec{G}^+ \right] + \left[ \frac{\mu}{\beta_r} \vec{G}^+ \right] \odot \left[ \frac{1}{\beta_r} \vec{G}^+ \right] \right) \\ &\quad + \left( \left[ \frac{\epsilon}{\alpha_r} \vec{G}^- \right] \odot \left[ \frac{1}{\alpha_r} \vec{G}^- \right] + \left[ \frac{\mu}{\beta_r} \vec{G}^- \right] \odot \left[ \frac{1}{\beta_r} \vec{G}^- \right] \right) \\ &\quad + \left( \left[ \frac{\epsilon}{\alpha_r} \vec{G}^+ \right] \odot \left[ \frac{1}{\alpha_r} \vec{G}^- \right] - \left[ \frac{\mu}{\beta_r} \vec{G}^+ \right] \odot \left[ \frac{1}{\beta_r} \vec{G}^- \right] \right) \\ &\quad + \left( \left[ \frac{\epsilon}{\alpha_r} \vec{G}^- \right] \odot \left[ \frac{1}{\alpha_r} \vec{G}^+ \right] - \left[ \frac{\mu}{\beta_r} \vec{G}^- \right] \odot \left[ \frac{1}{\beta_r} \vec{G}^+ \right] \right) \end{aligned} \quad (8.1.18)$$

$$\begin{aligned} \text{time domain. } \Rightarrow 8\mathcal{U}(t) &= \left( \left[ \frac{\epsilon}{\alpha_r} * \vec{G}^+ \right] \cdot \left[ \frac{1}{\alpha_r} * \vec{G}^+ \right] + \left[ \frac{\mu}{\beta_r} * \vec{G}^+ \right] \cdot \left[ \frac{1}{\beta_r} * \vec{G}^+ \right] \right) \\ &\quad + \left( \left[ \frac{\epsilon}{\alpha_r} * \vec{G}^- \right] \cdot \left[ \frac{1}{\alpha_r} * \vec{G}^- \right] + \left[ \frac{\mu}{\beta_r} * \vec{G}^- \right] \cdot \left[ \frac{1}{\beta_r} * \vec{G}^- \right] \right) \\ &\quad + \left( \left[ \frac{\epsilon}{\alpha_r} * \vec{G}^+ \right] \cdot \left[ \frac{1}{\alpha_r} * \vec{G}^- \right] - \left[ \frac{\mu}{\beta_r} * \vec{G}^+ \right] \cdot \left[ \frac{1}{\beta_r} * \vec{G}^- \right] \right) \\ &\quad + \left( \left[ \frac{\epsilon}{\alpha_r} * \vec{G}^- \right] \cdot \left[ \frac{1}{\alpha_r} * \vec{G}^+ \right] - \left[ \frac{\mu}{\beta_r} * \vec{G}^- \right] \cdot \left[ \frac{1}{\beta_r} * \vec{G}^+ \right] \right). \end{aligned} \quad (8.1.19)$$

The first two lines of eqn.(8.1.19) are the  $G^{+2}$  and  $G^{-2}$  terms that might naturally be expected to occur; the final two lines are  $G^+ \cdot G^-$  terms caused by interference between the two fields. In the dispersionless reference case, eqn.(8.1.19) reduces to

$$\begin{aligned} \text{freq. domain. } \Rightarrow 8\mathcal{U}(\omega) &= \left( \left[ \frac{\epsilon}{\epsilon_r} + \frac{\mu}{\mu_r} \right] \vec{G}^+ \right) \odot \vec{G}^+ + \left( \left[ \frac{\epsilon}{\epsilon_r} + \frac{\mu}{\mu_r} \right] \vec{G}^- \right) \odot \vec{G}^- \\ &\quad + \left( \left[ \frac{\epsilon}{\epsilon_r} - \frac{\mu}{\mu_r} \right] \vec{G}^+ \right) \odot \vec{G}^- + \left( \left[ \frac{\epsilon}{\epsilon_r} - \frac{\mu}{\mu_r} \right] \vec{G}^- \right) \odot \vec{G}^+ \end{aligned} \quad (8.1.20)$$

$$\begin{aligned} \text{time domain. } \Rightarrow 8\mathcal{U}(t) &= \left( \left[ \frac{\epsilon}{\epsilon_r} + \frac{\mu}{\mu_r} \right] * \vec{G}^+ \right) \cdot \vec{G}^+ + \left( \left[ \frac{\epsilon}{\epsilon_r} + \frac{\mu}{\mu_r} \right] * \vec{G}^- \right) \cdot \vec{G}^- \\ &\quad + \left( \left[ \frac{\epsilon}{\epsilon_r} - \frac{\mu}{\mu_r} \right] * \vec{G}^+ \right) \cdot \vec{G}^- + \left( \left[ \frac{\epsilon}{\epsilon_r} - \frac{\mu}{\mu_r} \right] * \vec{G}^- \right) \cdot \vec{G}^+. \end{aligned} \quad (8.1.21)$$



In the fully dispersionless case, eqn.(8.1.19) reduces to

$$\text{freq. domain.} \Rightarrow \quad 8 \mathcal{U}(\omega) = \left[ \frac{\epsilon}{\epsilon_r} + \frac{\mu}{\mu_r} \right] \vec{G}^+ \odot \vec{G}^+ + \left[ \frac{\epsilon}{\epsilon_r} + \frac{\mu}{\mu_r} \right] \vec{G}^- \odot \vec{G}^- \\ + \left[ \frac{\epsilon}{\epsilon_r} - \frac{\mu}{\mu_r} \right] \vec{G}^+ \odot \vec{G}^- + \left[ \frac{\epsilon}{\epsilon_r} - \frac{\mu}{\mu_r} \right] \vec{G}^- \odot \vec{G}^+ \quad (8.1.22)$$

$$\text{time domain.} \Rightarrow \quad 8 \mathcal{U}(t) = \left[ \frac{\epsilon}{\epsilon_r} + \frac{\mu}{\mu_r} \right] \vec{G}^+ \cdot \vec{G}^+ + \left[ \frac{\epsilon}{\epsilon_r} + \frac{\mu}{\mu_r} \right] \vec{G}^- \cdot \vec{G}^- \\ + 2 \left[ \frac{\epsilon}{\epsilon_r} - \frac{\mu}{\mu_r} \right] \vec{G}^+ \cdot \vec{G}^- . \quad (8.1.23)$$

We see here that the energy density takes a more complicated form if we choose to use reference parameters that do not match the true medium ones. If the reference parameters have a different degree of mismatch between  $\epsilon$  and  $\mu$ , which will in fact normally be the case, then the cross terms dependent on  $\vec{G}^+ \cdot \vec{G}^-$  do not vanish.

### 3. Full Vector Case

The energy density in the fully vectorised case can be found by retaining the  $[\vec{B} \cdot \vec{u}] [\vec{u} \cdot \vec{A}]$  corrections removed by the transverse approximation. Thus the non-dispersive reference energy density, as specified by eqn.(8.1.21), becomes

$$8 \mathcal{U} = \left( \left[ \frac{\epsilon}{\epsilon_r} + \frac{\mu}{\mu_r} \right] * \vec{G}^+ \right) \cdot \vec{G}^+ + \left( \left[ \frac{\epsilon}{\epsilon_r} + \frac{\mu}{\mu_r} \right] * \vec{G}^- \right) \cdot \vec{G}^- \\ \left( \left[ \frac{\epsilon}{\epsilon_r} - \frac{\mu}{\mu_r} \right] * \vec{G}^+ \right) \cdot \vec{G}^- + \left( \left[ \frac{\epsilon}{\epsilon_r} - \frac{\mu}{\mu_r} \right] * \vec{G}^- \right) \cdot \vec{G}^+ \\ + \frac{\mu}{\mu_r} * \left\{ \left[ \vec{G}^+ \cdot \vec{u} \right] \left[ \vec{u} \cdot \vec{G}^+ \right] + \left[ \vec{G}^- \cdot \vec{u} \right] \left[ \vec{u} \cdot \vec{G}^- \right] - \left[ \vec{G}^+ \cdot \vec{u} \right] \left[ \vec{u} \cdot \vec{G}^- \right] - \left[ \vec{G}^- \cdot \vec{u} \right] \left[ \vec{u} \cdot \vec{G}^+ \right] \right\} \quad (8.1.24)$$

The fully dispersive version of the corrections is

$$8 \mathcal{U}_{NT} = + \left[ \left( \frac{\mu}{\beta_r} * \vec{G}^+ \right) \cdot \vec{u} \right] \left[ \vec{u} \cdot \left( \frac{1}{\beta_r} * \vec{G}^+ \right) \right] + \left[ \left( \frac{\mu}{\beta_r} * \vec{G}^- \right) \cdot \vec{u} \right] \left[ \vec{u} \cdot \left( \frac{1}{\beta_r} * \vec{G}^- \right) \right] \\ - \left[ \left( \frac{\mu}{\beta_r} * \vec{G}^+ \right) \cdot \vec{u} \right] \left[ \vec{u} \cdot \left( \frac{1}{\beta_r} * \vec{G}^- \right) \right] + \left[ \left( \frac{\mu}{\beta_r} * \vec{G}^- \right) \cdot \vec{u} \right] \left[ \vec{u} \cdot \left( \frac{1}{\beta_r} * \vec{G}^+ \right) \right]. \quad (8.1.25)$$

### B. Poynting vector

The Poynting vector defines the energy flux, and is

$$\vec{P} = \vec{E} \times \vec{H} \longrightarrow P_z = E_x H_y \quad (8.2.1)$$

We will see that a  $G^+$  component always corresponds to a Poynting vector (energy flux) in the  $+z$  direction, and the  $G^-$  to an flux in the opposite direction (i.e.  $-z$ ). This is the basis of the assertion that these  $G^\pm$  variable are direction, and thus correspond to forward and backward directed field components.

**NOTE:** A critique of Poynting vectors – see e.g W. Gough, “*Poynting in the wrong direction*”, Eur. J. Phys. **3**, 83 (1982).



## 1. Vector Identities

Because the convolution integrals are independent of the vector properties, we can work in the transverse  $\vec{G}^\pm$  limit to get

$$-\vec{G}^+ \otimes \vec{u} \times \vec{G}^- = -[\vec{G}^+ \odot \vec{G}^-] \vec{u} + [\vec{G}^+ \cdot \vec{u}] * \vec{G}^- \quad (8.2.2)$$

$$= -[\vec{G}^+ \odot \vec{G}^-] \vec{u} \quad (8.2.3)$$

$$\text{and } \vec{G}^- \otimes \vec{u} \times \vec{G}^+ = [\vec{G}^- \odot \vec{G}^+] \vec{u} - [\vec{G}^- \cdot \vec{u}] * \vec{G}^+ \quad (8.2.4)$$

$$= +[\vec{G}^- \odot \vec{G}^+] \vec{u} \quad (8.2.5)$$

## 2. Transverse Case

The Poynting vector defines the energy flux. To get a simple expression, we need to work in the transverse case and specialise to a non-dispersive reference medium. Some expressions remain in the weaker partially transverse “THF” case – see III C, where  $\vec{u} \cdot \vec{H} = 0$  and  $2\beta_r \vec{H} = \vec{u} \times (\vec{G}^+ - \vec{G}^-)$ . Starting from the standard expression, and making use of the notation in eqn.(8.0.5)), we get

$$\vec{S}(t) = \vec{E}(t) \times \vec{H}(t) \quad (8.2.6)$$

$$\text{frequency domain } \Rightarrow \vec{S}(\omega) = \int d\omega' \vec{E}(\omega') \times \vec{H}(\omega - \omega'); \quad (8.2.7)$$

$$= \vec{E}(\omega) \otimes \vec{H}(\omega); \quad (8.2.8)$$

$$\text{THF approx. } \Rightarrow = \left\{ \left( \frac{1}{2\alpha_r} \right) [\vec{G}^+ + \vec{G}^-] \right\} \otimes \left\{ [\vec{u} \times \vec{G}^+ - \vec{u} \times \vec{G}^-] \left( \frac{-1}{2\sqrt{\mu_r}} \right) \right\} \quad (8.2.9)$$

$$-4 \vec{S} = \vec{G}_\epsilon^+ \otimes [\vec{u} \times \vec{G}_\mu^+] - \vec{G}_\epsilon^+ \otimes [\vec{u} \times \vec{G}_\mu^-] + \vec{G}_\epsilon^- \otimes [\vec{u} \times \vec{G}_\mu^+] - \vec{G}_\epsilon^- \otimes [\vec{u} \times \vec{G}_\mu^-]. \quad (8.10)$$

$$\text{simplify as below (VIII B 1); } = \vec{G}_\epsilon^+ \otimes [\vec{u} \times \vec{G}^+] - \vec{G}_\epsilon^- \otimes [\vec{u} \times \vec{G}^-] \quad (8.2.11)$$

$$= [\vec{G}_\epsilon^+ \odot \vec{G}_\mu^+] \vec{u} - [\vec{G}_\epsilon^+ \cdot \vec{u}] * \vec{G}_\mu^+ - [\vec{G}_\epsilon^- \odot \vec{G}_\mu^-] \vec{u} + [\vec{G}_\epsilon^- \cdot \vec{u}] * \vec{G}_\mu^- \quad (8.2.12)$$

$$= [\vec{G}_\epsilon^+ \odot \vec{G}_\mu^+] \vec{u} - 0 - [\vec{G}_\epsilon^- \odot \vec{G}_\mu^-] \vec{u} + 0 \quad (8.2.13)$$

$$= [\vec{G}_\epsilon^+ \odot \vec{G}_\mu^+ - \vec{G}_\epsilon^- \odot \vec{G}_\mu^-] \vec{u} \quad (8.2.14)$$

$$= [(\alpha_r^{-1} \vec{G}^+) \odot (\beta_r^{-1} \vec{G}^+) - (\alpha_r^{-1} \vec{G}^-) \odot (\beta_r^{-1} \vec{G}^-)] \vec{u} \quad (8.2.15)$$

$$\text{time domain; } -4\vec{S} = [(\alpha_r^{-1} * \vec{G}^+) \cdot (\beta_r^{-1} * \vec{G}^+) - (\alpha_r^{-1} * \vec{G}^-) \cdot (\beta_r^{-1} * \vec{G}^-)] \vec{u} \quad (8.2.16)$$



## 3. Minkowski Version

Starting from the standard expression, I use the shorthand notation  $\vec{G}_{M\epsilon}^{\pm} = \vec{G}^{\pm}\epsilon.\epsilon_r^{-1/2}$  and  $\vec{G}_{M\mu}^{\pm} = \vec{G}^{\pm}\mu.\mu_r^{-1/2}$  to avoid cluttered equations; and so

$$\vec{S}_M(t) = \vec{D}(t) \times \vec{B}(t) \quad (8.2.17)$$

$$= \epsilon * \vec{E}(t) \times \mu * \vec{H}(t) \quad (8.2.18)$$

$$\text{frequency domain } \Rightarrow \quad \vec{S}_M(\omega) = \int d\omega' \epsilon(\omega') \vec{E}(\omega') \times \mu(\omega - \omega') * \vec{H}(\omega - \omega'); \quad (8.2.19)$$

$$= \epsilon \vec{E} \otimes \mu \vec{H}(\omega); \quad (8.2.20)$$

$$\text{THF approx. } \Rightarrow \quad = \left\{ \left( \frac{\epsilon}{2\alpha_r} \right) [\vec{G}^+ + \vec{G}^-] \right\} \otimes \left\{ [\vec{u} \times \vec{G}^+ - \vec{u} \times \vec{G}^-] \left( \frac{-\mu}{2\sqrt{\mu_r}} \right) \right\} \quad (8.2.21)$$

$$-4 \vec{S}_M = \vec{G}_{M\epsilon}^+ \otimes [\vec{u} \times \vec{G}_{M\mu}^+] - \vec{G}_{M\epsilon}^+ \otimes [\vec{u} \times \vec{G}_{M\mu}^-] + \vec{G}_{M\epsilon}^- \otimes [\vec{u} \times \vec{G}_{M\mu}^+] - \vec{G}_{M\epsilon}^- \otimes [\vec{u} \times \vec{G}_{M\mu}^-] \quad (8.2.22)$$

$$\text{simplify as below (VIII B 1);} \quad = \vec{G}_{M\epsilon}^+ \otimes [\vec{u} \times \vec{G}^+] - \vec{G}_{M\epsilon}^- \otimes [\vec{u} \times \vec{G}^-] \quad (8.2.23)$$

$$= [\vec{G}_{M\epsilon}^+ \odot \vec{G}_{M\mu}^+] \vec{u} - [\vec{G}_{M\epsilon}^+ \cdot \vec{u}] * \vec{G}_{M\mu}^+ - [\vec{G}_{M\epsilon}^- \odot \vec{G}_{M\mu}^-] \vec{u} + [\vec{G}_{M\epsilon}^- \cdot \vec{u}] * \vec{G}_{M\mu}^- \quad (8.2.24)$$

$$= [\vec{G}_{M\epsilon}^+ \odot \vec{G}_{M\mu}^+] \vec{u} - 0 - [\vec{G}_{M\epsilon}^- \odot \vec{G}_{M\mu}^-] \vec{u} + 0 \quad (8.2.25)$$

$$= [\vec{G}_{M\epsilon}^+ \odot \vec{G}_{M\mu}^+ - \vec{G}_{M\epsilon}^- \odot \vec{G}_{M\mu}^-] \vec{u} \quad (8.2.26)$$

$$= \left[ \left( \frac{\epsilon}{\alpha_r} \vec{G}^+ \right) \odot \left( \frac{\mu}{\beta_r} \vec{G}^+ \right) - \left( \frac{\epsilon}{\alpha_r} \vec{G}^- \right) \odot \left( \frac{\mu}{\beta_r} \vec{G}^- \right) \right] \vec{u} \quad (8.2.27)$$

$$\text{time domain; } \quad -4 \vec{S}_M = \left[ \left( \frac{\epsilon}{\alpha_r} * \vec{G}^+ \right) \cdot \left( \frac{\mu}{\beta_r} * \vec{G}^+ \right) - \left( \frac{\epsilon}{\alpha_r} * \vec{G}^- \right) \cdot \left( \frac{\mu}{\beta_r} * \vec{G}^- \right) \right] \vec{u} \quad (8.2.28)$$

## IX. INTERPRETATIONS

The most comprehensive discussions on the interpretation of  $G^{\pm}$  fields are contained in Kinsler et.al. 2005 [2]. Other important remarks on the behaviour of the propagating fields at interfaces are covered in my report “*Causality in spatially propagated optics*” (tb arXiv).



## X. SECOND ORDER EVOLUTION EQUATIONS

In sections IV, V, and VI I calculated various forms of first-order propagation equation for the  $G^\pm$  field variables. However, many pulse propagation theories start from a second-order form for  $E$ , as in e.g. [8],

$$(\partial_z^2 + \nabla_\perp^2) E(\vec{r}, t) - \frac{1}{c^2} \partial_t^2 \int_{-\infty}^t dt' \epsilon(t-t') E(\vec{r}, t') = \frac{4\pi}{c^2} \partial_t^2 P(\vec{r}, t). \quad (10.0.1)$$

To address this point of interest, here I follow a similar path and derive a second-order  $G^\pm$  propagation equation. As expected, it looks rather similar to the standard equivalents for  $E$ , but with some extra derivative (curl) terms. First I attempt to make eqns.(3.1.6,3.1.7) look similar to eqn.(10.0.1) in subsection X A. Then in subsection X B I present a more complete derivation of the vector form in the transverse field case.

### A. Scalar Form

This is intended as a quick, intuitive calculation to get a scalar second-order equation for  $G^\pm$  fields. It is based on the old Fleck[1] definitions, so I leave the linear dispersion (from the permittivity  $\epsilon$  or permeability  $\mu$ ) inside the polarization  $P$  rather than separate it out as an extra term (thus it contains a hidden convolution). This approach has been superceded (I now put everything in  $\epsilon, \mu$ ), but this part is rather old and has not been updated accordingly. To start, I multiply eqn.(3.1.6) by the LHS derivative term seen in eqn.(3.1.7):

$$[\partial_t - \partial_z] [\partial_t + \partial_z] G^+ = [\partial_t - \partial_z] [-\partial_t P - \sigma(G^+ + G^-)] \quad (10.1.1)$$

$$(A) \quad = -[\partial_t - \partial_z] \partial_t P - \sigma [\partial_t - \partial_z] G^+ - \sigma [\partial_t - \partial_z] G^- \quad (10.1.2)$$

$$(B: \text{sub (3.1.7) for last RHS term}) \quad = -[\partial_t - \partial_z] \partial_t P - \sigma [\partial_t - \partial_z] G^+ - \sigma [-\partial_t P - \sigma G^+ - \sigma G^-] \quad (10.1.3)$$

$$(C: \text{iterate}) \quad = -[\partial_t - \partial_z - \sigma] \partial_t P - \sigma [\partial_t - \partial_z - \sigma] G^+ + \sigma^2 G^- \quad (10.1.4)$$

$$(D) \quad = - \left[ \partial_t - \partial_z + \sum_{n=1}^N (-\sigma)^n \right] \partial_t P - \sigma \left[ \partial_t - \partial_z + \sum_{n=1}^N (-\sigma)^n \right] G^+ + \sigma^{N+1} G^- \quad (10.1.5)$$

$$= -[\partial_t - \partial_z + Q] \partial_t P - \sigma [\partial_t - \partial_z + Q] G^+ + \sigma^{N+1} G^-, \quad (10.1.6)$$

$$Q = \frac{1 - \sigma^N}{1 - \sigma}, \text{ for } \sigma < 1, \quad (10.1.7)$$

It would now be straightforward to introduce a variety of useful approximations, e.g.  $\sigma \ll 1$  or similar. Alternatively, I could group the  $\sigma$  terms with the derivative operators in eqns.(3.1.6,3.1.7). I rearrange eqns.(3.1.6,3.1.7) to give

$$[\partial_t + \partial_z + \sigma] G^+ = -\partial_t P - \sigma G^-, \quad (10.1.8)$$

$$[\partial_t - \partial_z + \sigma] G^- = -\partial_t P - \sigma G^+, \quad (10.1.9)$$

and apply the differential operator from the LHS of eqn.(10.1.9) to eqn.(10.1.8)

$$[\partial_t - \partial_z + \sigma] [\partial_t + \partial_z + \sigma] G^+ = [\partial_t - \partial_z + \sigma] [-\partial_t P - \sigma G^-], \quad (10.1.10)$$

$$= -[\partial_t - \partial_z + \sigma] \partial_t P - \sigma [-\partial_t P - \sigma G^+] \quad (10.1.11)$$

$$= -[\partial_t - \partial_z] \partial_t P + \sigma^2 G^+ \quad (10.1.12)$$

$$\{[\partial_t - \partial_z + \sigma] [\partial_t + \partial_z + \sigma] - \sigma^2\} G^+ = -[\partial_t - \partial_z] \partial_t P. \quad (10.1.13)$$

Now I simplify the term in braces  $\{...\}$ , but retain the ordering of  $\sigma$ 's and  $\partial$ , because  $\sigma$  contains  $\epsilon$  – I might like to make the medium dispersive, even though it is customary to add dispersion at a later stage.

$$\{[\partial_t - \partial_z + \sigma] [\partial_t + \partial_z + \sigma] - \sigma^2\} G^+ \quad (10.1.14)$$

$$\{[(\partial_t + \sigma) - \partial_z] [(\partial_t + \sigma) + \partial_z] - \sigma^2\} G^+ \quad (10.1.15)$$

$$\{(\partial_t + \sigma)^2 + (\partial_t + \sigma) \partial_z - \partial_z (\partial_t + \sigma) - \partial_z^2 - \sigma^2\} G^+ \quad (10.1.16)$$

$$\{(\partial_t + \sigma)^2 + \partial_t \partial_z - \partial_z \partial_t + \sigma \partial_z - \partial_z \sigma - \partial_z^2 - \sigma^2\} G^+ \quad (10.1.17)$$

$$\{(\partial_t + \sigma)^2 + (\sigma \partial_z - \partial_z \sigma) - \partial_z^2 - \sigma^2\} G^+ \quad (10.1.18)$$

$$\{\partial_t^2 + \sigma \partial_t + \partial_t \sigma + (\sigma \partial_z - \partial_z \sigma) - \partial_z^2\} G^+. \quad (10.1.19)$$



And for  $G^-$ , the term would be  $\{\partial_t^2 + \sigma\partial_t + \partial_t\sigma - (\sigma\partial_z - \partial_z\sigma) - \partial_z^2\} G^-$ . Apply this to the full equation, assert the usual case of a uniform material (so  $\sigma\partial_z G^\pm = \partial_z\sigma G^\pm$ ), and then duplicate the calculation for  $G^-$  ...

$$\{\partial_z^2 - \partial_t^2 - \sigma\partial_t - \partial_t\sigma\} G^+ = [\partial_t - \partial_z] \partial_t P, \quad (10.1.20)$$

$$\{\partial_z^2 - \partial_t^2 - \sigma\partial_t - \partial_t\sigma\} G^- = [\partial_t + \partial_z] \partial_t P, \quad (10.1.21)$$

Compare this to the “standard” second order wave equation eqn.(10.0.1): (1) the  $\partial_z^2 G^\pm$  is analogous to  $\partial_z^2 E$ ; (2)  $(\partial_t + \partial_z) \partial_t P$  has an extra  $\partial_z \partial_t P$  part which might be expected given the directional nature of  $G^\pm$ ; (3) the  $-(\partial_t^2 + \sigma\partial_t + \partial_t\sigma) G^\pm$  is the analogous term to  $-\partial_t^2 \epsilon E dt$ ; remember that  $\sigma$  is a conductivity term not usually included in pulse propagation problems. If we add and subtract eqns.(10.1.20) and (10.1.21), setting  $\sigma = 0$ , we get

$$\{\partial_z^2 - \partial_t^2\} (G^+ + G^-) = \partial_t^2 P, \quad (10.1.22)$$

$$\{\partial_z^2 - \partial_t^2\} (G^+ - G^-) = \partial_z \partial_t P. \quad (10.1.23)$$

Of course  $E_x \propto G^+ + G^-$ , and  $H_y \propto G^+ - G^-$ , so the equivalence between eqn.(10.1.22) and eqn.(10.0.1) is to be expected. Presumably there would also be a related equivalence between eqn.(10.1.23) and the  $H_y$  version of eqn.(10.0.1).

## B. Vector Form

This is a vectorised derivation of the second order wave equation for  $\vec{G}^\pm$  fields. Unlike the previous scalar derivation, it is reasonably complete and without stringent approximation (barring its transverse field approx). The polarization term  $P$  remains as part of the dispersion term, thus simplifying the equations; until the end when I split it off for illustrative purposes. From section III, the unit vector in the propagation direction is  $\vec{u}$ , and the vectorised version of  $G$  is given by eqn.(3.3.1) –

$$\vec{G}^\pm = \tilde{\alpha}_r \vec{E} + \vec{u} \times \tilde{\beta}_r \vec{H}. \quad (10.2.1)$$

In frequency space,

$$-\imath\omega (\tilde{\epsilon} \vec{E}) = -\imath\omega (\tilde{\alpha}_r^2 + \tilde{\alpha}_r \tilde{\alpha}_c) \vec{E} = \nabla \times \vec{H} - \vec{J}, \quad (10.2.2)$$

$$-\imath\omega (\tilde{\mu} * \vec{H}) = -\imath\omega (\tilde{\beta}_r^2 + \tilde{\beta}_r \tilde{\beta}_c) \vec{H} = -\nabla \times \vec{E}. \quad (10.2.3)$$

In the following calculation, I omit the convolution symbols “\*” that should appear between the  $\tilde{\alpha}_c, \tilde{\beta}_c$  parameters and the  $G^+ \pm G^-$  fields; this is in the interests of both my laziness and reducing notational clutter. I split (as before in subsection II) the permittivity and permeability into instantaneous and time-dependent parts with  $\epsilon(\omega) = \alpha(\omega)^2 = \alpha_r^2 + \alpha_r \alpha_c(\omega)$  (and similarly for  $\mu$  and  $\beta$ ). This mimics traditional derivations, where dispersion and nonlinear polarization are handled separately. Using  $\vec{u} \times \nabla \times \vec{Q} - \nabla (\vec{u} \cdot \vec{Q}) = \nabla \times \vec{u} \times \vec{Q}$ , I take the scaled time derivative of the definition of  $\vec{G}^\pm$ ,

$$(-\imath\omega) \alpha_r \beta_r \vec{G}^\pm = -\imath\omega \alpha_r^2 \beta_r \vec{E} \pm (-\imath\omega) \alpha_r \beta_r^2 \vec{u} \times \vec{H} \quad (10.2.4)$$

$$= \nabla \times \beta_r \vec{H} - \beta_r \vec{J} + \imath\omega \alpha_c \alpha_r \vec{E} \mp \vec{u} \times [\nabla \times \alpha_r \vec{E}] \pm \imath\omega \vec{u} \times \beta_r \vec{H} \quad (10.2.5)$$

$$= \nabla \times \beta_r \vec{H} - \beta_r \vec{J} + \imath\omega \alpha_c \alpha_r \vec{E} \mp \nabla \times [\vec{u} \times \alpha_r \vec{E}] \mp \nabla [\vec{u} \cdot \alpha_r \vec{E}] \pm \imath\omega \vec{u} \times \beta_r \vec{H} \quad (10.2.6)$$

$$= \nabla \times \beta_r \vec{H} - \beta_r \vec{J} \mp \nabla \times [\vec{u} \times \alpha_r \vec{E}] \mp \nabla [\vec{u} \cdot \alpha_r \vec{E}] + \imath\omega \alpha_c \alpha_r \vec{E} \pm \imath\omega \vec{u} \times \beta_r \vec{H} \quad (10.2.7)$$

$$= \mp \nabla \times [\vec{u} \times \alpha_r \vec{E} \mp \beta_r \vec{H}] - \beta_r \vec{J} \mp \nabla [\vec{u} \cdot \alpha_r \vec{E}] + \imath\omega \alpha_c \alpha_r \vec{E} \pm \imath\omega \vec{u} \times \beta_r \vec{H} \quad (10.2.8)$$

$$= \mp \nabla \times [\vec{u} \times \alpha_r \vec{E} \pm \vec{u} \times \vec{u} \times \beta_r \vec{H}] \pm \nabla \times (\vec{u} \cdot \beta_r \vec{H}) \vec{u} - \beta_r \vec{J} \mp \nabla [\vec{u} \cdot \alpha_r \vec{E}] \\ + \imath\omega \alpha_c \alpha_r \vec{E} \pm \imath\omega \vec{u} \times \beta_r \vec{H}. \quad (10.2.9)$$



Now I combine  $\vec{E}$ 's and  $\vec{H}$ 's to form  $\vec{G}^\pm$ 's,

$$-\imath\omega\alpha_r\beta_r\vec{G}^\pm = \mp\nabla\times[\vec{u}\times\vec{G}^\pm] \pm G^0\nabla\times\vec{u} \mp\nabla[\vec{u}\cdot\alpha_r\vec{E}] + \imath\omega\alpha_c\alpha_r\vec{E} \pm \imath\omega\vec{u}\times\beta_c\beta_r\vec{H} - \beta_r\vec{J} \quad (10.2.10)$$

$$= \mp\nabla\times[\vec{u}\times\vec{G}^\pm] + \frac{\imath\omega}{2}\alpha_c[\vec{G}^+ + \vec{G}^-] \pm \frac{\imath\omega}{2}\beta_c[\vec{G}^+ - \vec{G}^-] \mp\frac{1}{2}\nabla[\vec{u}\cdot(\vec{G}^+ + \vec{G}^-)] - (\text{B2.11})$$

$$= \mp\vec{u}\times[\nabla\times\vec{G}^\pm] \pm\nabla[\vec{u}\cdot\vec{G}^\pm] + \frac{\imath\omega}{2}\alpha_c[\vec{G}^+ + \vec{G}^-] \pm \frac{\imath\omega}{2}\beta_c[\vec{G}^+ - \vec{G}^-] \mp\frac{1}{2}\nabla[\vec{u}\cdot(\vec{G}^+ + \vec{G}^-)] - \beta_r\vec{J} \quad (10.2.12)$$

$$= \mp\vec{u}\times[\nabla\times\vec{G}^\pm] + \frac{\imath\omega}{2}\alpha_c[\vec{G}^+ + \vec{G}^-] \pm \frac{\imath\omega}{2}\beta_c[\vec{G}^+ - \vec{G}^-] + \frac{1}{2}\nabla[\vec{u}\cdot(\vec{G}^+ - \vec{G}^-)] - (\text{B2.13})$$

and apply another scaled time derivative  $-\imath\omega\alpha_r\beta_r$ ,

$$\begin{aligned} -\omega^2\alpha_r^2\beta_r^2\vec{G}^\pm &= \pm\imath\omega\alpha_r\beta_r\vec{u}\times\nabla\times\vec{G}^\pm + \frac{\omega^2}{2}\alpha_r\beta_r\alpha_c[\vec{G}^+ + \vec{G}^-] \mp\frac{\omega^2}{2}\alpha_r\beta_r\beta_c[\vec{G}^+ - \vec{G}^-] \\ &\quad - \frac{\imath\omega}{2}\alpha_r\beta_r\nabla[\vec{u}\cdot(\vec{G}^+ - \vec{G}^-)] + \imath\omega\alpha_r\beta_r^2\vec{J} \end{aligned} \quad (10.2.14)$$

$$\begin{aligned} (\text{B}) \quad &= \mp\vec{u}\times\nabla\times(-\imath\omega\alpha_r\beta_r\vec{G}^\pm) + \frac{\omega^2}{2}\alpha_r\beta_r\alpha_c[\vec{G}^+ + \vec{G}^-] \mp\frac{\omega^2}{2}\alpha_r\beta_r\beta_c[\vec{G}^+ - \vec{G}^-] \\ &\quad - \frac{\imath\omega}{2}\alpha_r\beta_r\nabla[\vec{u}\cdot(\vec{G}^+ - \vec{G}^-)] + \imath\omega\alpha_r\beta_r^2\vec{J} \end{aligned} \quad (10.2.15)$$

$$\begin{aligned} (\text{C}) \quad &= (\mp\vec{u}\times\nabla\times)\left\{\mp\vec{u}\times[\nabla\times\vec{G}^\pm] + \frac{\imath\omega}{2}\alpha_c[\vec{G}^+ + \vec{G}^-] \pm \frac{\imath\omega}{2}\beta_c[\vec{G}^+ - \vec{G}^-] + \frac{1}{2}\nabla[\vec{u}\cdot(\vec{G}^+ - \vec{G}^-)] - \beta_r\vec{J}\right\} \\ &\quad + \frac{\omega^2}{2}\alpha_r\beta_r\alpha_c[\vec{G}^+ + \vec{G}^-] \mp\frac{\omega^2}{2}\alpha_r\beta_r\beta_c[\vec{G}^+ - \vec{G}^-] - \frac{\imath\omega}{2}\alpha_r\beta_r\nabla[\vec{u}\cdot(\vec{G}^+ - \vec{G}^-)] + \imath\omega\alpha_r\beta_r^2\vec{J} \end{aligned} \quad (10.2.16)$$

$$\begin{aligned} (\text{D}) \quad &= \vec{u}\times\nabla\times\vec{u}\times[\nabla\times\vec{G}^\pm] \mp\frac{\imath\omega}{2}\alpha_c\vec{u}\times\nabla\times[\vec{G}^+ + \vec{G}^-] - \frac{\imath\omega}{2}\beta_c\vec{u}\times\nabla\times[\vec{G}^+ - \vec{G}^-] \\ &\quad \mp\frac{1}{2}\nabla\vec{u}\times\nabla\times[\vec{u}\cdot(\vec{G}^+ - \vec{G}^-)] \pm\vec{u}\times\nabla\times\beta_r\vec{J} \\ &\quad + \frac{\omega^2}{2}\alpha_r\beta_r\alpha_c[\vec{G}^+ + \vec{G}^-] \mp\frac{\omega^2}{2}\alpha_r\beta_r\beta_c[\vec{G}^+ - \vec{G}^-] - \frac{\imath\omega}{2}\alpha_r\beta_r\nabla[\vec{u}\cdot(\vec{G}^+ - \vec{G}^-)] + \imath\omega\alpha_r\beta_r^2\vec{J} \end{aligned} \quad (10.2.17)$$

$$\begin{aligned} (\text{E}) \quad &= \vec{u}\times\nabla\times\vec{u}\times[\nabla\times\vec{G}^\pm] + \frac{\omega^2}{2}\alpha_r\beta_r\alpha_c[\vec{G}^+ + \vec{G}^-] \mp\frac{\imath\omega}{2}\alpha_c\vec{u}\times\nabla\times[\vec{G}^+ + \vec{G}^-] \\ &\quad \mp\frac{\omega^2}{2}\alpha_r\beta_r\beta_c[\vec{G}^+ - \vec{G}^-] - \frac{\imath\omega}{2}\beta_c\vec{u}\times\nabla\times[\vec{G}^+ - \vec{G}^-] \\ &\quad \mp\frac{1}{2}\nabla\vec{u}\times\nabla\times[\vec{u}\cdot(\vec{G}^+ - \vec{G}^-)] - \frac{\imath\omega}{2}\alpha_r\beta_r\nabla[\vec{u}\cdot(\vec{G}^+ - \vec{G}^-)] + \imath\omega\alpha_r\beta_r^2\vec{J} \pm\vec{u}\times\nabla\times\beta_r\vec{J} \end{aligned} \quad (10.2.18)$$

$$\begin{aligned} (\text{F}) \quad &= \vec{u}\times\nabla\times\vec{u}\times[\nabla\times\vec{G}^\pm] - (-\imath\omega)\frac{1}{2}\left[\frac{1}{c_r}(-\imath\omega) \mp\vec{u}\times\nabla\times\right]\alpha_c[\vec{G}^+ + \vec{G}^-] \\ &\quad \pm(-\imath\omega)\frac{1}{2}\left[\frac{1}{c_r}(-\imath\omega) \mp\vec{u}\times\nabla\times\right]\beta_c[\vec{G}^+ - \vec{G}^-] \\ &\quad + \frac{1}{2}\nabla\left[\frac{1}{c_r}(-\imath\omega) \mp\vec{u}\times\nabla\times\right]\vec{u}\cdot[\vec{G}^+ - \vec{G}^-] - \left[\frac{1}{c_r}(-\imath\omega) \mp\vec{u}\times\nabla\times\right]\beta_r\vec{J} \end{aligned} \quad (10.2.19)$$

$$\begin{aligned} (\text{G}) \quad &= \vec{u}\times\nabla\times\vec{u}\times[\nabla\times\vec{G}^\pm] \\ &\quad -(-\imath\omega)\frac{1}{2}\left[\frac{1}{c_r}(-\imath\omega) \mp\vec{u}\times\nabla\times\right]\left\{\alpha_c[\vec{G}^+ + \vec{G}^-] \pm\beta_c[\vec{G}^+ - \vec{G}^-] + \beta_r\vec{J}\right\} \\ &\quad + \frac{1}{2}\nabla\left[\frac{1}{c_r}(-\imath\omega) \mp\vec{u}\times\nabla\times\right]\vec{u}\cdot[\vec{G}^+ - \vec{G}^-] \end{aligned} \quad (10.2.20)$$



Time domain (non-dispersive reference) –

$$\begin{aligned} \frac{d^2}{dt^2} G^\pm &= \vec{u} \times \nabla \times \vec{u} \times [\nabla \times \vec{G}^\pm] \\ &\quad - \frac{1}{2} \frac{d}{dt} \left[ \frac{1}{c_r} \frac{d}{dt} \mp \vec{u} \times \nabla \times \right] \left\{ \alpha_c * [\vec{G}^+ + \vec{G}^-] \pm \beta_c * [\vec{G}^+ - \vec{G}^-] + \beta_r \vec{J} \right\} \\ &\quad + \frac{1}{2} \nabla \left[ \frac{1}{c_r} \frac{d}{dt} \mp \vec{u} \times \nabla \times \right] \vec{u} \cdot [\vec{G}^+ - \vec{G}^-]. \end{aligned} \quad (10.2.21)$$

Since I've already assumed transverse fields, in the source-free case  $\nabla \cdot \vec{E} = 0$  so  $\nabla \times \nabla \times \vec{Q} \rightarrow \nabla^2 \vec{Q}$ . Thus,

$$\nabla^2 \vec{G}^\pm - \frac{1}{c^2} \frac{d^2}{dt^2} \vec{G}^\pm - \frac{1}{2} \frac{d}{dt} \left\{ \frac{1}{c_r} \frac{d}{dt} \mp \vec{u} \times \nabla \times \right\} \left\{ \alpha_c * [\vec{G}^+ + \vec{G}^-] \pm \beta_c * [\vec{G}^+ - \vec{G}^-] \right\} = 0. \quad (10.2.22)$$

Note the strong similarity to eqns.(10.1.20,10.1.21). The usual polarization term  $\vec{P}$  is contained in the  $\alpha_c$  (and  $\beta_c$ ) terms. If I write  $\alpha_c = \alpha_c^D + \alpha_c^{NL}$  and  $\vec{P} = \alpha_r \alpha_c^{NL} \vec{E}$ ; dropping the magnetic terms (i.e.  $\beta_c = 0$ ) gives us:

$$\nabla^2 \vec{G}^\pm - \frac{1}{c^2} \frac{d^2}{dt^2} \vec{G}^\pm - \frac{1}{2} \frac{d}{dt} \left\{ \frac{1}{c_r} \frac{d}{dt} \mp \vec{u} \times \nabla \times \right\} \alpha_c^D * [\vec{G}^+ + \vec{G}^-] = + \frac{1}{2\alpha_r} \frac{d}{dt} \left\{ \frac{1}{c_r} \frac{d}{dt} \mp \vec{u} \times \nabla \times \right\} \vec{P}. \quad (10.2.23)$$

The  $1/\alpha_r$  prefactor occurs on the RHS because we are relating  $G^\pm$  to the polarization  $P$ , rather than the usual  $E$  and  $P$ .

We could use these second order equations to apply diffraction to a standard first-order equation propagation, by split-stepping the diffraction given here  $\nabla^2 \vec{G}^\pm$  with the usual dispersive and nonlinear propagation.

### C. GFEA Form

My “Few cycle pulse propagation” detailed calculation [9] has included terms for  $G^\pm$  field envelopes; where the GFEA equation is ( $g = \pm 1$ ):

$$\begin{aligned} q \partial_\xi A(\vec{r}_\perp, \xi, \tau) &= - \frac{\sigma \mp \sigma'}{1 \pm i\sigma' \partial_\tau} \partial_\tau A(\vec{r}_\perp, \xi, \tau) + \frac{1 + i\sigma \partial_\tau}{1 \pm i\sigma' \partial_\tau} \left( -\frac{\alpha_r}{\beta_r} + i\hat{D}' \right) A(\vec{r}_\perp, \xi, \tau) + \frac{i}{2\beta_r^2 (1 \pm i\sigma' \partial_\tau)} \nabla_\perp^2 A(\vec{r}_\perp, \xi, \tau) \\ &\quad + \frac{2i\pi}{n_0^2} [1 \pm i\partial_\tau \oplus i\sigma' \partial_\tau \oplus g (1 - iq\partial_\xi)] \frac{(1 \pm i\partial_\tau)}{(1 \pm i\sigma' \partial_\tau)} B(\vec{r}_\perp, \xi, \tau; A) + \frac{T_\Gamma + T_{RHS}}{1 \pm i\sigma' \partial_\tau}. \end{aligned} \quad (10.3.1)$$

Here the  $\oplus$  and  $\mp$  symbols act like the usual  $\pm, \mp$  ones; but varying according to which  $G$  variable is being described – i.e.  $G \mp$ . The normal  $\pm, \mp$  ones are relate to the carrier direction.

## XI. ENVELOPE PROPAGATION EQUATIONS

It is common in relatively narrowband cases to solve for the propagation of field envelopes centred around chosen carrier frequencies rather than the full detailed oscillations of the EM field. In fact, if sufficient care is taken with approximations, and if the system simulated is well behaved, even quite wideband pulses can be sucessfully modelled using envelopes. The advantage of the  $G^\pm$  wave equations presented in sections IV, V, and VI is that we dispense with the second order form of the wave equation, and generate envelope equations from first order equations. This greatly reduces the number of approximations without increasing the complexity of the theory. Although a full model requires four envelopes to describe the  $G^\pm$ , just as a full Maxwell theory requires four envelopes, (both backward and forward travelling  $E$  and  $H$ ), this is rarely a case we are interested in solving, and so in practice we can use just one envelope.

A full expansion of  $G^\pm$  into forward and backward envelopes  $G_f^\pm, G_b^\pm$  would be

$$\begin{aligned} G^\pm(\omega) &= G_f^\pm(\omega \mp \omega_0) e^{\pm ik_0 z} + G_f^\pm(\omega_0 \pm \omega) e^{\mp ik_0 z} \\ &\quad + G_b^\pm(\omega \mp \omega_0) e^{\mp ik_0 z} + G_b^\pm(\omega_0 \mp \omega) e^{\pm ik_0 z}, \end{aligned} \quad (11.0.1)$$



where we have suppressed the  $z$  argument on the envelope functions for brevity. Note that the argument shift ( $\omega \rightarrow \omega \pm \omega_0$ ) is how the  $-\imath\omega_0 t$  part of the carrier wave is accounted for. If inserted into the wave equations, this expansion would result in a large number of terms to consider, even for the relatively simple case of a third order nonlinearity. However, by specializing to the typical case where we are only interested in forward travelling waves, we can instead use

$$G^\pm(\omega) = \mathcal{G}_f^\pm(\omega \mp \omega_0)e^{\pm\imath k_0 z} + \mathcal{G}_f^\pm(\omega_0 \pm \omega)e^{\mp\imath k_0 z} \quad (11.0.2)$$

$$= \mathcal{G}_f^\pm(\omega \mp \omega_0)e^{\pm\imath k_0 z} + \mathcal{G}_f^\pm(\omega \mp \omega_0)^*e^{\mp\imath k_0 z}, \quad (11.0.3)$$

$$G^\pm(t) = \mathcal{G}_f^\pm(t)e^{\pm\imath k_0 z}e^{\mp\imath\omega_0 t} + \mathcal{G}_f^\pm(t)^*e^{\mp\imath k_0 z}e^{\pm\imath\omega_0 t} \quad (11.0.4)$$

This has the nice advantage that it eliminates and backward travelling waves, and so we can propagate pulses efficiently in a moving frame. This is important, because the backward parts in a moving frame move at *twice* the frame speed. In a full (non-envelope) simulation, we need to either take special efforts to filter out these backward parts, or any hoped-for numerical gains are reversed by the finer  $z$ -step required to integrate (the backward parts) accurately.

### A. Foward travelling envelopes in a first order wave equation

I start with the plane polarized first order wave eqn. (4.2.29)

$$\partial_{z'} G_x^\pm = \mp\imath\omega\alpha_r\beta_r(1 \mp \xi) G_x^\pm \mp \frac{\imath\omega\beta_r}{2}\alpha_c * [G_x^+ + G_x^-] - \frac{\imath\omega\alpha_r}{2}\beta_c * [G_x^+ - G_x^-]. \quad (11.1.1)$$

I insert envelopes using the shorthand notation  $\Xi = \imath k_0 z$ . Since  $\alpha_c$  may contain a dependence on the fields  $G^\pm$ , I split it using  $\alpha_c = a_c + a_c^*$ . In the case of dispersion,  $a_c = \alpha_c/2$ ; but for a nonlinearity it will be the appropriately carrier-matched, positive frequency part of  $\alpha_c$ , once the field values it contains have been expanded in terms of the envelope and carrier functions. I will proceed in the stationary frame case ( $\xi = 0$ ), so

$$\begin{aligned} \partial_z [\mathcal{G}_f^\pm e^{\pm\Xi} + \mathcal{G}_f^\pm * e^{\mp\Xi}] &= \mp\imath\omega\alpha_r\beta_r [\mathcal{G}_f^\pm e^{\pm\Xi} + \mathcal{G}_f^\pm * e^{\mp\Xi}] \\ &\mp \frac{\imath\omega(a_c + a_c^*)\beta_r}{2} \{ \mathcal{G}_f^\pm e^{\pm\Xi} + \mathcal{G}_f^\pm * e^{\mp\Xi} + \mathcal{G}_f^\mp e^{\mp\Xi} + \mathcal{G}_f^\mp * e^{\pm\Xi} \}. \end{aligned} \quad (11.2)$$

$$\begin{aligned} e^{\pm\Xi} \partial_z \mathcal{G}_f^\pm \pm \imath k_0 e^{\pm\Xi} \mathcal{G}_f^\pm + e^{\mp\Xi} \partial_z \mathcal{G}_f^\pm * \mp \imath k_0 e^{\mp\Xi} \mathcal{G}_f^\pm * &= \mp\imath\omega\alpha_r\beta_r [\mathcal{G}_f^\pm e^{\pm\Xi} + \mathcal{G}_f^\pm * e^{\mp\Xi}] \\ &\mp \frac{\imath\omega(a_c + a_c^*)\beta_r}{2} \{ \mathcal{G}_f^\pm e^{\pm\Xi} + \mathcal{G}_f^\pm * e^{\mp\Xi} + \mathcal{G}_f^\mp e^{\mp\Xi} + \mathcal{G}_f^\mp * e^{\pm\Xi} \}. \end{aligned} \quad (11.3)$$

$$\text{split c.c. halves; } e^{\pm\Xi} \partial_{z'} \mathcal{G}_f^\pm \pm \imath k_0 e^{\pm\Xi} \mathcal{G}_f^\pm = \mp\imath\omega\alpha_r\beta_r \mathcal{G}_f^\pm e^{\pm\Xi} \mp \frac{\imath\omega a_c \beta_r}{2} \{ \mathcal{G}_f^\pm e^{\pm\Xi} + \mathcal{G}_f^\mp * e^{\pm\Xi} \} \quad (11.1.4)$$

$$\text{cancel } e^{\pm\Xi}; \quad \partial_z \mathcal{G}_f^\pm \pm \imath k_0 \mathcal{G}_f^\pm = \mp\imath\omega\alpha_r\beta_r \mathcal{G}_f^\pm \mp \frac{\imath\omega a_c \beta_r}{2} \{ \mathcal{G}_f^\pm + \mathcal{G}_f^\mp * \} \quad (11.1.5)$$

$$\partial_z \mathcal{G}_f^\pm = \mp\imath(\omega\alpha_r\beta_r - k_0) \mathcal{G}_f^\pm \mp \frac{\imath\omega a_c \beta_r}{2} \{ \mathcal{G}_f^\pm + \mathcal{G}_f^\mp * \}, \quad (11.1.6)$$

If we do the sensible thing and relate the carrier parameters  $k_0$  and  $\omega_0$  using the reference parameters (so  $k_0 = \omega_0/c_r(\omega_0)$ ), we have  $\omega\alpha_r\beta_r - k_0 = (\omega - \omega_0)\alpha_r\beta_r + \omega_0\alpha_r\beta_r - k_0 = (\omega - \omega_0)\alpha_r\beta_r$ :

$$\partial_z \mathcal{G}_f^\pm = \mp\imath(\omega - \omega_0) \alpha_r\beta_r \mathcal{G}_f^\pm \mp \frac{\imath\omega a_c \beta_r}{2} \{ \mathcal{G}_f^\pm + \mathcal{G}_f^\mp * \}, \quad (11.1.7)$$

Now transform into the moving frame. Note that we will no longer want the frame to move at the phase velocity at  $\omega_0$ , because that job has been taken over by the carrier; but we could use it to absorb any residual (reference) dispersion affecting one of the envelopes (most likely for  $\mathcal{G}_f^+$ ). However, it will then make the residual (reference) dispersion term acting on the other envelope (e.g.  $\mathcal{G}_f^-$ ) twice as large. Anyway

$$\partial_{z'} \mathcal{G}_f^\pm - \imath\omega\alpha_f\beta_f \mathcal{G}_f^\pm = \mp\imath(\omega\alpha_r\beta_r - k_0) \mathcal{G}_f^\pm \mp \frac{\imath\omega a_c \beta_r}{2} \{ \mathcal{G}_f^\pm + \mathcal{G}_f^\mp * \}, \quad (11.1.8)$$

$$\implies \partial_{z'} \mathcal{G}_f^\pm = \mp\imath(\omega\alpha_r\beta_r \mp \imath\omega\alpha_f\beta_f - k_0) \mathcal{G}_f^\pm \mp \frac{\imath\omega a_c \beta_r}{2} \{ \mathcal{G}_f^\pm + \mathcal{G}_f^\mp * \} \quad (11.1.9)$$

**B. Envelopes and the Second Order wave equation**

See [9], which includes correction terms appropriate to  $G^\pm$  fields. Note that there I can think of no reason to use the second order form, since the first order form performs as well with fewer complications.

**XII. VECTOR IDENTITIES**

(1) Identity (old Physics 31.320 vector identity sheet, (I-5)):

$$\nabla \left( \vec{u} \cdot \vec{H} \right) = (\vec{u} \cdot \nabla) \vec{H} + (\vec{H} \cdot \nabla) \vec{u} + \vec{u} \times (\nabla \times \vec{H}) + \vec{H} \times (\nabla \times \vec{u}) \quad (12.0.1)$$

$$\text{rearrange} \quad \vec{u} \times (\nabla \times \vec{H}) = \nabla \left( \vec{u} \cdot \vec{H} \right) - (\vec{u} \cdot \nabla) \vec{H} - (\vec{H} \cdot \nabla) \vec{u} - \vec{H} \times (\nabla \times \vec{u}) \quad (12.0.2)$$

$$\text{constant unit vector} \quad = \nabla \left( \vec{u} \cdot \vec{H} \right) - (\vec{u} \cdot \nabla) \vec{H} - 0 - 0 \quad (12.0.3)$$

$$\text{transverse field} \quad = 0 - (\vec{u} \cdot \nabla) \vec{H} - 0 - 0 \quad (12.0.4)$$

$$\text{Useful: } (\vec{u} \cdot \nabla) \vec{H} = \nabla \left( \vec{u} \cdot \vec{H} \right) - \vec{u} \times (\nabla \times \vec{H}) \quad (12.0.5)$$

(2) Identity (old Physics 31.320 vector identity sheet, (I-10)):

$$\nabla \times (\vec{u} \times \vec{H}) = \vec{u} \cdot (\nabla \cdot \vec{H}) - \vec{H} \cdot (\nabla \cdot \vec{u}) + (\vec{H} \cdot \nabla) \vec{u} - (\vec{u} \cdot \nabla) \vec{H} \quad (12.0.6)$$

$$\text{constant unit vector} \quad = \vec{u} \cdot (\nabla \cdot \vec{H}) - 0 + 0 - (\vec{u} \cdot \nabla) \vec{H} \quad (12.0.7)$$

$$\text{no monopoles} \quad = 0 - 0 + 0 - (\vec{u} \cdot \nabla) \vec{H} \quad (12.0.8)$$

$$\text{Useful: } (\vec{u} \cdot \nabla) \vec{H} = -\nabla \times (\vec{u} \times \vec{H}). \quad (12.0.9)$$

These two identities, specialised to the case involving our unit vector  $\vec{u}$  in the propagation direction, mean that we can equate

$$\vec{u} \times (\nabla \times \vec{H}) - \nabla \left( \vec{u} \cdot \vec{H} \right) = \nabla \times (\vec{u} \times \vec{H}) \quad (12.0.10)$$

Note also

$$\vec{a} \times \vec{b} \times \vec{c} = (\vec{a} \cdot \vec{c}) \vec{b} - (\vec{a} \cdot \vec{b}) \vec{c} \quad (12.0.11)$$

so in our case where  $\vec{a}$  and  $\vec{b}$  are both unit vectors

$$\vec{u} \times (\vec{u} \times \vec{X}) = (\vec{u} \cdot \vec{X}) \vec{u} - (\vec{u} \cdot \vec{u}) \vec{X} \quad (12.0.12)$$

Also, I use (old Physics 31.320 vector identity sheet, (I-7)):

$$\nabla \cdot (\vec{A} \times \vec{B}) = \vec{A} \cdot (\nabla \times \vec{B}) - \vec{B} \cdot (\nabla \times \vec{A}) \quad (12.0.13)$$



## XIII. UNITS

$$[\mu] = N/A^2 = m \cdot kg \cdot s^{-2} \cdot A^{-2} \quad (13.0.1)$$

$$[B] = V \cdot s \cdot m^{-1} \quad (13.0.2)$$

$$[\mu H^2] = [\mu (B/\mu)^2] = [B^2/\mu] = V^2 \cdot s^2 \cdot m^{-2} \cdot m^{-1} \cdot kg^{-1} \cdot s^2 \cdot A^2 \quad (13.0.3)$$

$$= [V \cdot A \cdot s^2 \cdot m^{-2}]^2 \cdot m^{-1} \cdot kg^{-1} \quad (13.0.4)$$

$$[\epsilon] = F/m = A^2 \cdot s^4 \cdot m^{-3} \cdot kg^{-1} \quad (13.0.5)$$

$$[E] = V \cdot m^{-1} \quad (13.0.6)$$

$$[\epsilon E^2] = A^2 \cdot s^4 \cdot m^{-3} \cdot kg^{-1} \cdot V^2 \cdot m^{-2} \quad (13.0.7)$$

$$= [V \cdot A \cdot s^2 \cdot m^{-2}]^2 \cdot m^{-1} \cdot kg^{-1} \quad (13.0.8)$$